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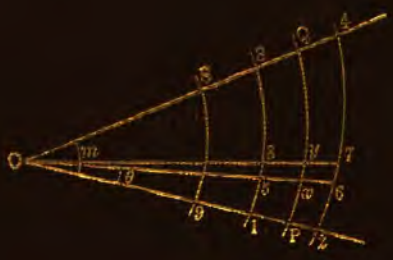

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PRISMOIDAL FORMULÆ

AND

EARTHWORK.

BY

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INTRODUCTION.

THE following definitions will be used not only on account of their historic importance, but also on account of their adoption in the literature of mathematics. As far as possible these definitions have been taken *verbatim* from their original authors.

Prismoid.—Charles Hutton in his work upon mensuration (1770), published at Newcastle-upon-Tyne, gave the following definition: "A prismoid is a solid having for its ends any dissimilar plane figures of the same number of sides, and all the sides of the solid plane figures. It is evident that the sides of this solid are all trapezoids. If the ends of the prismoid be bounded by curves, as ellipses, etc., the number of its sides or trapezoids will be infinite, and it is then called, sometimes, a cylindroid." In brief, then, Hutton's definition is that a prismoid is a solid whose parallel bases are polygons of the same number of sides and whose side faces are trapezoids.

The term *prismatoid* was introduced by Wittstein in 1860 in "Das Prismatoid, eine Erweiterung der elementaren Stereometrie." It was defined as a polyhedron with any two parallel polygonal bases, and whose side faces are triangles formed by joining the vertices of the bases, so that each line in order forms a triangle with the preceding line and one side of either base. The prismatoid is thus a special case of the prismoid, i.e., when one side of the side trapezoid becomes zero.

When the bases are closed curves in parallel planes and the mantle is generated by a straight line that moves on the two curves as directors and which finally returns to its initial position from the opposite side from which it started, it generates a ruled surface which can be called a cylindroid.

As the theorem of Koppe will be referred to, we append here his definition of a prismoid (obelisk): "A body which is determined by two parallel polygons as bases and by trapezoids as side faces shall receive the name of *obelisk*."

If a line passes through a fixed point in the plane of one of the bases of the prismoid (cylindroid) and moves so that it is always

parallel to the generating line of the prismoid (cylindroid), it will cut out between the bases a pyramid (cone) which is called the *associate* pyramid (cone).

The mean area of a prismoid is an area which being multiplied by the height (or altitude) gives the volume.

It would be sufficient to group all these solids (prismoids, prismatoids) under the general name of prismoid. When we use the term *prismoidal* formula it carries with it the general meaning.

The honor of having demonstrated the rule for finding solidity of the prismoid has been ascribed to Thomas Simpson (1750) by some writers; but in reality it is due to Newton ("Methodus Differentialis," 1711). Newton stated that the volume could be found by multiplying the sum of the bases and four times the mid-section by one-sixth of the altitude when any cross-section is a cubic function of the distance from the section from which the volume is reckoned. Of course this applies to all solids whose cross-sections are quadratic, linear, or constant functions of their distances from the reference section.

It is probable that the confusion that has arisen as to the priority of discovery of the prismoidal formula is due to the fact that Simpson's rule is similar in form and equivalent in process of derivation to the prismoidal formula. Both formulæ are for finding the sum of all values of any variable (y) that is a quadratic function of some other variable (x) between certain limits, a and b, of x.

In 1770 Charles Hutton, in his work alluded to above, gave another demonstration of the formula. Hutton called this his *general rule*, and he also derived a *particular rule*. The latter he stated virtually as follows: Let a and b be the sides of one rectangular base; c and d, the sides of the other rectangular base; h, its altitude, and v, its volume; then

$$v = \frac{h}{6}\Big[(2a + c)b + (2c + a)d\Big].*$$

In commenting on the scope and applicability of the general rule, he remarks that "this rule will serve to find the volume of any prismoid or cylindroid of whatever figure the end may be." Later in his work (p. 457) he shows that this general rule applies to the frusta of all solids generated by the revolution of a conic section about a straight line and to all solids whose sections are similar figures. This statement was made by Simpson twenty years before.

The next advance was made by the discovery and proof of what we call the *associate* formula. The priority of discovery of this formula belongs to Meier Hirsch, who published it in his "Sammlung

* Ellwood Morris, Measurement of Earthwork, 1871.

Geometrischer Aufgaben" in 1807. The theorem as enunciated by Hirsch is as follows: The contents of every body that is circumscribed by two parallel bases of an equal number of sides and by as many oblique (sloping) quadrilaterals as each of the bases has sides is equal to the remainder which will be obtained if from the product of the sum of the two bases and the half altitude the sum of the products of each two successive projections of its side lines into the sines of its positive or negative angles of incidence by the twelfth part of the altitude be subtracted.

The advent of the railroad acted as a spur upon the active genius of mankind in every avocation, and every device for shortening and lightening labor was welcomed. It had been fully realized that the prismoidal formula was the only formula known that would give the exact contents of the earthwork solid by using sections of the prismoid itself, and Sir John MacNeill's Tables (1833) that were based upon it were gladly received as another time-saver.

While the Newtonian formula had been established by analytic methods, Steiner (in 1842) demonstrated by elementary geometry its applicability to the prismoid in its widest sense. Four years before (1838), Karl Koppe published his celebrated formula in Crelle's Journal. In (1843) this "New Theorem of Stereometry" was issued in a small pamphlet of 38 pages.

In the Journal of the Franklin Institute, December 1857, p. 372, W. M. Gillespie showed that the Newtonian formula applies to the volume of earthwork when the upper surface is generated by a straight line moving on two longitudinal straight lines, thus making the upper surface a hyperbolic paraboloid. This proof was based on elementary principles, but it had really been given long before. It can readily be shown that any cross-section is a quadratic function of its distance from either base, and therefore the Newtonian formula applies. Still the method of Gillespie's proof was elegant, and it has been widely copied.

In 1864 Mehler in his "Elementar Mathematik," page 121, established the two-term formula where the volume is found by multiplying one-fourth of the altitude by the sum of either base and three times a section at two-thirds the altitude from such base. The method of his proof is given in Chap. I. This two-term formula had been published by Hermann Kinklin (1862).

The following is a collection of the principal prismoidal formulæ, with dates of publication:

1. Newton's formula, published in 1711: $M = \frac{1}{6}(B_1 + 4S_{\frac{1}{2}} + B_2)$;
2. Hirsch's " " " 1807: $M = \frac{1}{2}(B_1 + B_2) - \frac{1}{6}A$;
3. Koppe's " " " 1838: $M = S_{\frac{1}{2}} + \frac{1}{12}A$;
4. Kinklin's " " " 1862: $M = \frac{1}{4}(B_1 + 3T)$;
5. Echol's " " " 1894: $M = \frac{1}{4}(S_x + S_y)$;

where $x = \frac{1}{6}(3 + \sqrt{3})$ and $y = \frac{1}{6}(3 - \sqrt{3})$.

6. The one-third formula, published in 1895: $M = \frac{1}{8}(S_x + S_{\frac{1}{2}} + S_y)$, where $x = \frac{1}{4}(2 + \sqrt{2})$ and $y = \frac{1}{4}(2 - \sqrt{2})$.

It is observed that there are three two-term and three three-term formulæ. Formulæ 1 and 5 have a wider and a higher application, as they give (as will be seen later) the volume of any solid between parallel planes whose sections are cubic functions of their distance from the reference section,

PREFACE.

THE object of this work is to present a ready method of estimating the usual quantities in earthwork computations by graphical methods and to confine these diagrams to as few as possible. The method here outlined will apply to the majority of earthwork calculations, whether the volume is estimated by the Newtonian or average end-area formula. As the latter method is only approximate even for prismatic solids, the "Earthwork Diagram" will give results well within the usual limits of accuracy.

Another object that has been kept in view was to discover the original author of the usual formulæ associated with the prismoid. It is believed that this has been done, and the honor here ascribed to Newton, Hirsch, Koppe, and Kinklin can be verified by history.

The chapter on two-term formulæ was presented as a thesis by the writer at Cornell University, in 1895.

THOS. U. TAYLOR.

AUSTIN, TEXAS, June, 1897.

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PRISMOIDAL FORMULÆ AND EARTHWORK.

CHAPTER I.

PRISMOIDAL FORMULÆ.

1. The volume of a prismatoid is equal to the *sum of the bases and four times the mid-section multiplied by one-sixth the altitude.*

Join any point P in the plane of the mid-section to the vertices of each polygonal base. The prismatoid is thus divided into pyramids of three classes all having their vertices at P The first class is

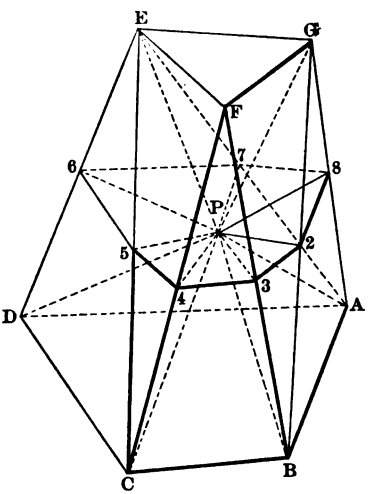

FIG. 1.

composed of *P-EFG*, having its base in the upper base of the prismatoid; the second class is composed of *P-ABCD*, having its base in the lower base of the prismatoid; and the third class is composed of those having a vertex at P and their bases the side faces *ECD*, *CEF*, etc.

Let M = mean area of prismatoid;
B_1 = area of lower base of prismatoid;
B_2 = area of upper base of prismatoid;
H = altitude of prismatoid;
and for convenience, $P56 = a$; $P67 = b$; $P78 = c$; $P82 = d$; $P23 = e$; $P34 = f$; and $P45 = g$.

(1) Volume $P\text{-}EFG = \frac{1}{3}$ base $\times$ altitude $= \frac{1}{6}B_2H$.

(2) Volume $P\text{-}ABCD = \frac{1}{3}$ base $\times$ altitude $= \frac{1}{6}B_1H$.

(3) Volume of $P\text{-}ECD = 4$ times volume $P\text{-}E56$, as they have their vertices at the same point, and triangle $EDC = 4$ times triangle $E56$.

But volume $P\text{-}E56 = \frac{1}{3}$ altitude $\times$ base $= \frac{1}{6}H \times E56 = \frac{1}{6}Ha$.

$\therefore$ volume $P\text{-}ECD = \frac{4}{6}Ha$.

In the same way volume $P\text{-}ECF = \frac{4}{6}Hg$; $P\text{-}BCF = \frac{4}{6}Hf$, etc.

$$\therefore \text{ volume of prismatoid} = \frac{H}{6}(B_1+B_2+4a+4b+4c+4d+4e+4f+4g).$$

But $a+b+c+d+e+f =$ mid-section of prismatoid $= S_{\frac{1}{2}}$.

$$\therefore V = \frac{H}{6}(B_1+4S_{\frac{1}{2}}+B_2, \quad \ldots\ldots\ldots \quad (1)$$

2. The volume of a cylindroid can be found by the formula just established. Inscribe in each base of the cylindroid a polygon of as many sides as possible. Join the vertices of these polygons so as to form a prismatoid. Call the bases and mid-section of the prismatoid B_1' B_2', and $S_{\frac{1}{2}}'$. Then if V' represents its volume, we have

$$V' = \frac{H}{6}(B_1'+4S_{\frac{1}{2}}'+B_2').$$

Let M' = mean area of prismatoid. $\therefore V' = HM'$.

As the number of sides of the polygon increases indefinitely, the bases and mid-section of the prismatoid approach the bases and mid-sections of the cylindroid as their limits. The volume (V') approaches the volume (V) of the cylindroid as its limit. When two variables maintain a constant ratio their limits have the same ratio; hence we have

$$V' \div M' = H, \text{ a constant};$$
$$\therefore V \div M = H = V' \div M'.$$
$$\therefore \text{ limit } (V' \div M') = (V \div M) = H.$$

$$\therefore \text{ volume cylindroid} = \frac{H}{6}(B_1+4S_{\frac{1}{2}}+B_2).$$

3. Koppe's Theorem.*—*Every obelisk (prismoid) is equal to the sum of a prism and a pyramid each of which has an altitude equal to that of the obelisk and whose bases correspond in angles to the bases of the obelisk, while the sides of the base of the prism are the half-sums of the sides of the obelisk, and the sides of the base of the pyramid are half the differences of the corresponding sides of the bases.*

Let ABC-DEF be a triangular prismoid. Draw the mid-section 789, and through 7 and 8 draw 65 and 14 parallel to CF, and through 8 draw 283 parallel to AD. Join 12 and 34. The sides of the triangle 789 are equal to the half-sum of the sides to which they are

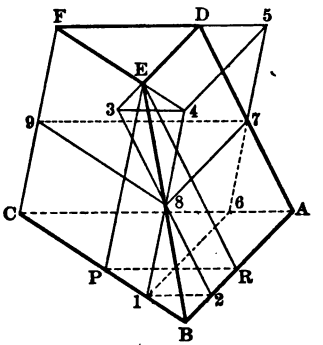

FIG. 2.

parallel, as they join in the mid-points of the sides of the trapezoids; the sides of the triangle $B12$ are equal to the half-difference of the sides to which they are parallel. Let the sides of the triangle ABC and DEF be indicated by small letters opposite the given angles.

Now $89 = \frac{1}{2}(a+d)$, $87 = \frac{1}{2}(c+f)$, $97 = \frac{1}{2}(e+b)$.

$B1 = a - \frac{1}{2}(a+d) = \frac{1}{2}(a-d)$, $B2 = \frac{1}{2}(c-f)$, $12 = \frac{1}{2}(b-e)$.

Volume ABC-$DEF = C16\text{-}54F + 8\text{-}B21 + 8\text{-}E34$.

But volume of prism $C16\text{-}54F$ = area $789 \times H$,

and volume of pyramid $8\text{-}B21$ = area $B21 \times \frac{1}{6}H$ = pyr. $8\text{-}E34$.

$$\therefore \text{ volume of prismoid} = (789)H + (B21)\frac{H}{6} \times 2$$

$$= (789)H + \frac{H}{3}(B21).$$

* Ein neuer Lehrsatz der Stereometrie. Von Karl Koppe.

Conceive the side faces generated by a line touching the perimeter of the polygons in such a manner that while one point on the line moves uniformly from E to F, the other moves from B to C; and so from F to D, and from D to E. If a line passes through E (or any other point in upper base) and moves so that it is always parallel to the generator, it will trace out a pyramid E-BPK, which is called the associate pyramid.

$$\text{Pyramid } E\text{-}BRP : 8\text{-}B21 :: \overline{BI}^3 : \overline{BP}^3 :: 8 : 1.$$

$$\therefore E\text{-}BRP = 8(8\text{-}B21).$$

Let A = base of associate pyramid = $4(B21) = PBR$.

$$\therefore E\text{-}BPR = \tfrac{1}{2}AH,$$

$$8\text{-}B21 = \tfrac{1}{24}AH.$$

$$\therefore \text{volume prismoid} = V = (789)H + \tfrac{1}{12}AH.$$

Let $S_{\frac{1}{2}}$ = mid section 789.

$$\therefore V = HS_{\frac{1}{2}} + \tfrac{1}{12}AH = H(S_{\frac{1}{2}} + \tfrac{1}{12}A).$$

$$\therefore \text{mean area } M = V \div H = S_{\frac{1}{2}} + \tfrac{1}{12}A. \quad . \quad . \quad . \quad (2)$$

Note that the perimeter of the mid-section equals one-half sum of perimeter of bases, and perimeter of mid-section of associate pyramid equals the half-difference of perimeter of bases.

4. The extension of the formula of Koppe to a prismoid having bases of any number of sides is easily accomplished. Let $ABCD$-$KGFE$, Fig. 3, be a prismoid. Extend any two sides of the upper base, as KG and EF, till they intersect in P, and the two sides respectively parallel to these in the lower base till they intersect in some point 5. Join $P5$. Now the volume of $ABCD$-$KGFE$ = volume of $A5D$-KPE less triangular pyramid $BC5$-PGF.

Draw the mid-section 6789, and produce 67 and 89 till they intersect $P5$ in Q. The mid sections of the triangular prisms $A5D$-KPE and $BC5$-GPF are $7Q8$ and $6Q9$. Through P draw lines $P1$, $P3$, $P2$, and $P4$ parallel respectively to EA, FB, KD, and GC. Call the points where these lines intersect the lower base, 1, 3, 2, and 4. Join 12 and 34. Now from the preceding proposition P-125 is the associate pyramid for the triangular prism $A5D$-KPE, and P-345 is the associate pyramid for the prism $B5C$-PGF. Therefore P-1234 is the associate pyramid for the prismoid $ABCD$-$KGFE$.

From the preceding proposition we have

$$\text{Volume } (A5D\text{-}EPK) = H[7Q8 + \tfrac{1}{12}(521)].$$

$$\text{Volume } (BC5\text{-}PFG) = H[69Q + \tfrac{1}{12}(534)].$$

Subtracting, we get

$$\text{Volume } ABCD\text{-}KGFE = H[6789 + \tfrac{1}{12}(1234)]$$

$$= H(S_{\frac{1}{2}} + \tfrac{1}{12}A).$$

In the same way we may extend the proposition to a prismoid whose bases have any number of sides whatever. Conceive the prismoid generated by a straight line starting at GC and moving clockwise from position GC to position GD, then from DG to DK, etc. Now suppose a line to pass through any point in the plane of

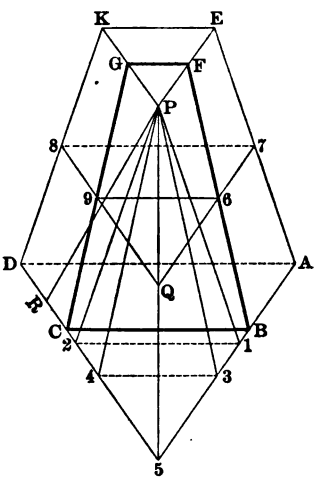

FIG. 3.

the upper base and to be drawn parallel to GC. It will in this position coincide with $P4$. Now let this line through P move so that it will be parallel to the generator of the prismoid. When this generator is at DG the line through P will coincide with PR, and as the generator DG moves to DK, the line through P will move from PR to $P2$. The resultant motion through P is 42. In reality it moves clockwise a distance $4R$ and contra-clockwise a distance $R2$, leaving the resultant motion = 42.

5. In 1838 Karl Koppe, Oberlehrer am Gymnasium zu Soest, published in Crelle's Journal the theorem that now bears his name. In 1843 it was republished in his "Ein neuer Lehrsatz der Stereometrie," a pamphlet that was intended to be a supplement to all text-books on Stereometry. The theorem as enunciated by Koppe was as follows:

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"Jeder Obelisk ist gleich der Summe aus einem Prisma und einer Pyramide, welche beide mit dem Obelisken gleiche Höhe haben, und deren Grundflächen in den Winkeln mit den Grundflächen des Obelisken übereinstimmen, während die Seiten der Grundfläche des Prismas den halben Summen und Seiten der Grundfläche der Pyramide den halben Differenzen der gleichliegenden Seiten der beiden Grundflächen des Obelisken gleich sind."

It is thus seen that Koppe's theorem as stated here is confined to the prismoid (obelisk). This theorem of Koppe's was easily established as above by elementary geometry for the old prismoid. The bases of the prism and pyramid were polygons whose perimeters were respectively the half-sum and the half-difference of the perimeters of the bases and whose angles were equal to the corresponding angles of the bases.

On page 20 of his "Neuer Lehrsatz" Koppe remarks: "The application of our proposition to bodies that are bounded (enclosed) by curved side faces does not confine itself to the frustum of a cone, but extends to all bodies that have two parallel bases and curved side faces that are generated by a straight line which is always on the curved lines that circumscribe the bases."

6. *The mean area of a prismatoid is equal to the half-sum of the bases less one-sixth of the base of the associate pyramid.*

Join any vertex, as *B* (Fig. 4.), with all the other vertices of the bases.

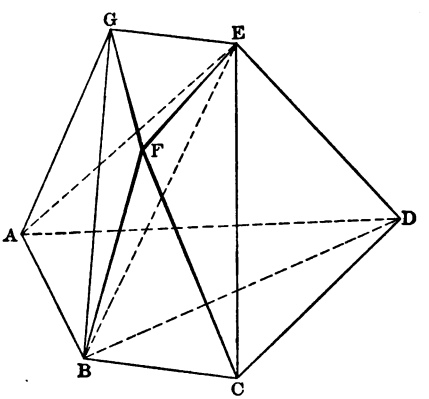

FIG. 4.

The prismatoid is composed of pyramid *B-EGF* (having base in upper base of prismatoid); the pyramids *E-BCD*, *E-ADB* (having their bases in the lower base of the prismatoid); and the tetrahedra *FE-BC* and *AB-GE*. There are thus three types of solids that

make up the prismatoid. We shall first show that the associate theorem applies to each class separately.

Given $V = MH$,

where M = mean area of prismatoid,
B_1 = area of lower base,
B_2 = area of the upper base,
A = area base of associate pyramid.

(*a*) In *B-EGF*, Fig. 5 (*a*), we have

$$V = \tfrac{1}{3}HB_2.$$

But by associate formula

$$V = H\left[\frac{B_1 + B_2}{2} - \frac{1}{6}A\right] = H[\tfrac{1}{2}B_2 - \tfrac{1}{6}B_2] = \tfrac{1}{3}HB_2.$$

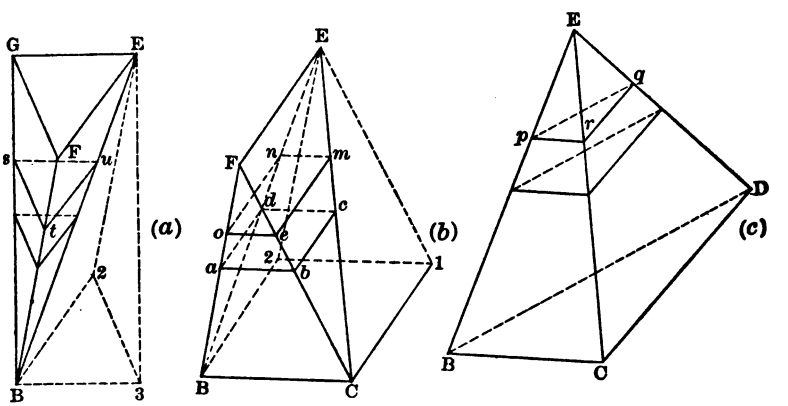

FIG. 5.

(*b*) In *FE-BC* we have, Fig. 5 (*b*)

$$V = \frac{H}{6}(B_1 + B_2 + 4S_{\frac{1}{2}}) = \frac{H}{6}(0 + 0 + 4S_{\frac{1}{2}}) = \tfrac{2}{3}HS_{\frac{1}{2}}.$$

But by the associate formula

$$V = H\left[\frac{B_1 + B_2}{2} - \frac{1}{6}A\right] = H[0 + 0 - \tfrac{1}{6}A] = \tfrac{1}{3}HS_{\frac{1}{2}}.$$

From Fig. 5 (*b*) we see that

$$BC12 = 4(abcd),$$

or $$A = 4S_{\frac{1}{2}}.$$

But the areas *abcd* and *BC*12 are described clockwise and contra-clockwise, or *vice versa*.

$$\therefore A = -4S_1.$$

(*c*) In *E-BCD* we have

$$V = \tfrac{1}{3}HB_1.$$

By the associate formula,

$$V = H\left[\frac{B_1 + B_2}{2} - \tfrac{1}{6}A\right]$$

$$= H\left[\frac{B_1}{2} + 0 - \tfrac{1}{6}B_1\right] = \tfrac{1}{3}HB_1.$$

The foregoing investigation is, perhaps, sufficient; but it will be well to follow out the investigation for the prismatoid as a whole. Let us assume that the mid-section of each primary solid of the prismatoid is as indicated in the second column of the following table:

Solid.	Mid-section.	Base of Associate Pyramid.	Upper Base.	Lower Base.	Volume.
B-EFG	*d*	$4d$	$4d$	0	$\frac{1}{3}Hd$
E-BCD	*e*	$4e$	0	$4e$	$\frac{1}{3}He$
E-ABD	*f*	$4f$	0	$4f$	$\frac{1}{3}Hf$
GE-BA	*g*	$-4g$	0	0	$\frac{1}{3}Hg$
FE-BC	*k*	$-4k$	0	0	$\frac{1}{3}Hk$

The base of the associate pyramid of the prismatoid is thus equal to the sum of the upper and lower bases of the prismatoid less the base of the associate pyramids of the tetrahedra that have only edges in the bases. It is necessary to show that the generator of the associate pyramid of the prismatoid graphically sums up these individual bases.

Fig. 6 shows the associate pyramid of the prismatoid of Fig. 4. The lower base is drawn complete for reference. By comparing the prismatoid and the associate pyramid we see that

*C*1 is equal and parallel to *FE*;
21 is equal and parallel to *BC*;
23 is equal and parallel to *FG*;
34 is equal and parallel to *AB*; and
4*A* is equal and parallel to *GE*.

The parallelograms $BC12$ and $B34A$ are the bases of associate pyramids of the tetrahedra $FE\text{-}BC$ and $GE\text{-}AB$. If we subtract these parallelograms from $ABCD$ and $B23$, there is left the base of the associate pyramid $ADC1234A$.

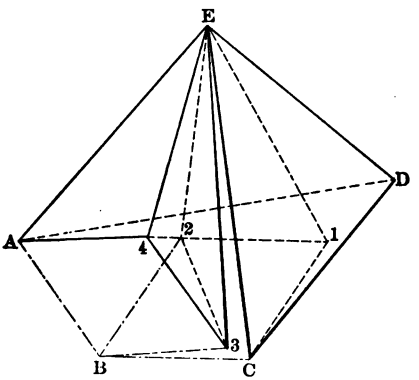

FIG. 6.

From the last column of the table we have

$$V = \frac{H}{6}[8d + 8e + 8f + 4g + 4k]$$
$$= \frac{H}{6}[12d + 12e + 12f - (4g + 4e + 4f - 4g - 4k)]$$
$$= \frac{H}{6}[3B_1 + 3B_2 - (4d + 4e + 4f - 4g - 4k)].$$

But from the third column,

$$A = 4d + 4e + 4f - 4g - 4k;$$

$$V = \frac{H}{6}(3B_1 + 3B_2 - A)$$
$$= H\left(\frac{B_1 + B_2}{2} - \frac{1}{6}A\right).$$

7. If in the base of the cylindroid we inscribe polygons, we can by joining the vertices of the different polygons with each other form a prismatoid. If the number of sides of these polygons be indefinitely increased, the bases of the prismatoid gradually approach the bases of the cylindroid as a limit, and the base of the associate pyramid approaches the base of the associate cone as a limit.

If V' = volume of prismatoid,
M' = mean area of prismatoid,
V = volume of cylindroid,
M = mean area of cylindroid,

then the ratio of V' to M' is always constant and equal to H; therefore their limits V and M have the same ratio.

$$\therefore \text{ limit of } \left[\frac{B_1 + B_2}{2} - \frac{1}{6}A\right] = \frac{B_1 + B_2}{2} - \frac{1}{6}A.$$

8. Koppe's formula can be established for the prismatoid by the same method we employed for the associate formula. We must bear in mind that the perimeters and areas of sections of the associate pyramid when described *contra* to those of the bases must be treated negatively.

Refer to Figs. 4, 5, and 6, and table in Art. 6.

(*a*) For *B-EFG*, Fig. 5 (*a*), we have

$$V = \tfrac{1}{3}HB_2.$$

By Koppe's formula,

$$V = H(S_{\frac{1}{2}} + \tfrac{1}{3}X) = H(d + \tfrac{1}{3}d) = \tfrac{4}{3}Hd = \tfrac{1}{3}HB_2.$$

(*b*) For *BC-EF* we have

$$V = \tfrac{1}{3}Hk.$$

By Koppe's formula,

$$V = H(S_{\frac{1}{2}} + \tfrac{1}{3}T) = H(k - \tfrac{1}{3}k) = \tfrac{2}{3}Hk.$$

(*c*) For *E-BCD* we have

$$V = \tfrac{1}{3}HB_1.$$

By Koppe's formula,

$$V = H(S_{\frac{1}{2}} + \tfrac{1}{3}T) = H(e + \tfrac{1}{3}e) = \tfrac{4}{3}Hf = \tfrac{1}{3}HB_1.$$

The application of the formula can be extended to the prismatoid as a whole. From the last column of the table we have

$$V = \frac{H}{6}(8d + 8e + 8f + 4g + 4k)$$
$$= \frac{H}{6}[(6d + 6e + 6f + 6g + 6k) + (2d + 2e + 2f - 2g - 2k)]$$
$$= \frac{H}{6}(6S + 2T) = H(S_{\frac{1}{2}} + \tfrac{1}{3}T).$$

Its extension to the cylindroid follows as in Art. 7.

9. The volume enclosed by a parallelogram $ACDE$, two triangles ACB and DCB, and a warped surface $ABDE$ that is generated by a straight line PR that moves from position DB to position EA in such a manner that the ends at the sects PR move simultaneously with uniform velocity, is equal to half the product of the base ABC by the altitude CD (Fig. 7).

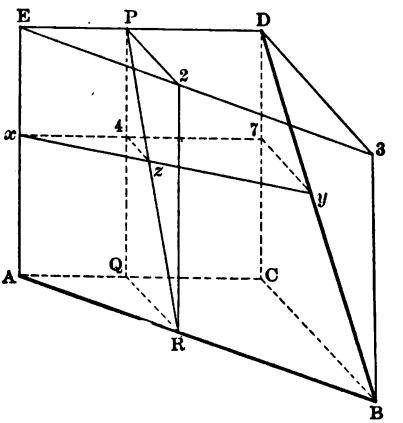

FIG. 7.

If PQ is drawn parallel to DC and QR joined, QR will be parallel to BC in every position. Complete the triangular prism ABC-$DE3$, and draw $R2$ parallel to $B3$, and join $P2$. $PQR2$ is a parallelogram. Now in every position of PR the section PQR of the solid is half of the section ($PQR2$) of the prism. The volume of the solid is therefore half of the volume of the prism.

$$\text{Volume } ABC\text{-}DE = \tfrac{1}{2}ABC \times \text{altitude } DC.$$

10. Hirsch's Theorem *.—*The volume determined by a parallelogram FECD, two triangles EBC and ADF, a quadrilateral ABCD, and a warped surface ABEF, generated as in Art. 9, is equal to the product of the altitude into the difference of half the base ABCD and one twelfth of the product of the projections of the side lines (AF and BE) and the sine of their angle of incidence.*

Let altitude $EC = H$, $ABCD = B_1$, $BCP = A$.

Then, by theorem, $V = H(\tfrac{1}{2}B_1 - \tfrac{1}{6}A)$.

* Sammlung geometrischer Aufgaben. Von Meier Hirsch. Zweiter Theil. Berlin, 1870.

Through A and C draw lines parallel to DC and DA respectively. Join PE. The whole solid is thus divided into a wedge $APCD\text{-}FE$;

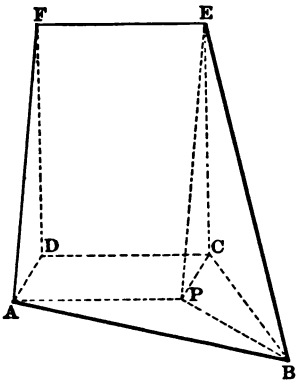

FIG. 8.

a triangular pyramid $E\text{-}PCB$, and a solid determined by parallelogram $APEF$, two triangles PEB and APB, and a warped surface $ABFE$.

For convenience let area $APCD = x$ and area $APB = y$.

Volume wedge $FE\text{-}PCAD = \frac{1}{3}H(APCD) = \frac{1}{3}Hx$;

Volume pyramid $E\text{-}PCB = \frac{1}{3}H(PCB) = \frac{1}{3}HA$;

Volume of $FE\text{-}PBA = \frac{1}{3}Hy$, (Art. 9.)

$$\therefore \text{total volume} = H(\tfrac{1}{3}x + \tfrac{1}{3}y + \tfrac{1}{3}A)$$
$$= H(\tfrac{1}{3}x + \tfrac{1}{3}y + \tfrac{1}{3}A - \tfrac{1}{6}A)$$
$$= H[(\tfrac{1}{3}x + y + A) - \tfrac{1}{6}A].$$

But $x + y + A = B_1$.

$$\therefore V = H(\tfrac{1}{3}B_1 - \tfrac{1}{6}A).$$

Conceive the volume generated by a straight line that moves from EB to EC, from EC to FD, from FD to AF, from AF to EB in such a manner that the generating line divides FE and AB proportionally etc. Now if we conceive a line which always passes through E, and which at first coincides with EB, to keep parallel to the generating line, we see that its trace on plane $ABCD$ will be the triangle CBP; that is, BCP is the base of the associate pyramid. The projections of the side lines are AD $(= PC)$ and BC. The product of

the projections of the side lines on the base into the sine of their angle of incidence gives double the base of the associate pyramid. That is,

$$PC \times BC \times \sin PCB = 2 \text{ area } PCB = 2A.$$

$$\therefore\ V = H[\tfrac{1}{2}B_1 - \tfrac{1}{12}PC \times BC \sin PCB].$$

11. In Fig. 9 let 13 and 24 be two positions of the generator near together. Project the upper base and point P on the lower base in 65 A and p, and from 1 and 2 drop perpendiculars 16 and 25 on base

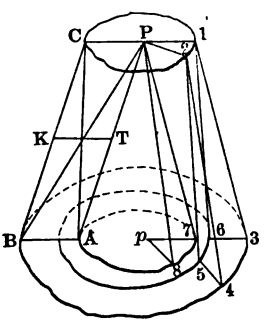

FIG. 9.

B_1. Join 36 and 45. Let $P7$ and $P8$ be generators of associate cone that are parallel to 13 and 24 respectively. Thus $p7$ and $p8$ are parallel and equal to 63 and 45.

Draw chords 12, 65, 34, and 78. The figure 1265 is a parallelogram. The volume of solid 12-3456 by former proposition is

$$H(\tfrac{1}{2} \text{ area } 3456 - \tfrac{1}{6} \text{ area } p78).$$

When 2 moves infinitesimally near 1 the chord 21 coincides with arc 12, and the same will be true of the other chords and arcs. Suppose that in the last position the lines $p7$ and $p8$ make a small angle θ, and let the area of 3456 be represented by x, and let $v =$ volume of the solid 12·3456. When 1 and 2 are near together we have

$$v = H[\tfrac{1}{2}x - \tfrac{1}{6}(\tfrac{1}{2}r^2\theta)]$$

$$= \tfrac{1}{2}Hx - \frac{H}{12}r^2\theta.$$

The volume generated by the triangle 245 as it moves from position 136 around to that position again will be Σv.

But $$V = \frac{H}{2}\Sigma(x) - \frac{H}{6}\Sigma_{\cdot}\, r\,\theta).$$

Let the lower base $34EF$ of the cylindroid $= B_1$, and the upper base

$$1C2 = 65A = B_2.$$

Now $$\Sigma v = \tfrac{1}{2}H\,\Sigma(x) - \frac{H}{6}[\Sigma_{\cdot} r^2\theta],$$

and $$\Sigma x = B_1 - B_2, \quad \Sigma(\tfrac{1}{2}r^2\theta) = A.$$

$$\therefore\ \Sigma(v) = \tfrac{1}{2}H(B_1 - B_2) - \tfrac{1}{6}HA.$$

The total volume will be composed of the cylinder and the solid generated by the variable triangle 245.

Volume cylinder $= H \times B_2$.

Volume solid generated by $245 = \Sigma v = \tfrac{1}{2}H(B_1 - B_2) - \tfrac{1}{6}HA$.

$$\therefore \text{ total volume of cylindroid } = HB_2 + \frac{H}{2}(B_1 - HB_2) - \tfrac{1}{6}HA$$

$$= H\left[\frac{(B_1 + B_2)}{2} - \frac{1}{6}A\right]. \quad . \quad . \quad . (3)$$

12. The warped surface $ABDE$, Fig. 7, may also be generated by a line that moves on DB and EA from position ED to position AB in time t, subject to the condition that the end points move with velocity of $\frac{EA}{t}$ and $\frac{DB}{t}$ respectively. Let a plane be passed through a point x one nth of the altitude from ED and parallel to ABC. This plane will divide all sects EA, PR, DC, and DB in ratio of 1 to $(n - 1)$. In other words, one nth of these sects will be above the cutting plane. This plane will intersect PR in some point z where PZ is one nth of PR. It will cut the plane of triangle PRQ in a line $Z4$ which is parallel to and one nth of QR. The plane will cut triangle BCD in line $y7$ which is parallel to and one nth of CB. The plane will cut parallelogram $ACDE$ in a line $x47$ parallel and equal to AC and ED. Therefore

$$Z4 : QR : : y7 : BC : : 1 : n,$$

or $$Z4 : y7 : : QR : BC.$$

But $$QR : BC : : AQ : AC : : x4 : x7.$$

$$\therefore\ Z4 : Y7 : : x4 : X7. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (Z)$$

Now as the points x, z, y, 4, and 7 all lie in one plane and have the relation (Z). and as $x47$ is a straight line, the points x, z, and y are also on a straight line. Thus all planes parallel to ABC will cut the warped surface in straight lines. The hyperbolic paraboloidal surface between four lines can thus be generated by a straight moving on either pair of opposite sides subject to the conditions of uniform velocity already established.

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13. *The volume of any prismatoid (cylindroid) is equal to a prism (cylinder) and a pyramid (cone), the altitude of each being equal to the altitude of the prismatoid (cylindroid), and their bases being respectively the mid-sections of the prismatoid (cylindroid) and the associate pyramid (cone).*

If the side faces of the solid are warped, the condition that the perimeters of the mid-sections of the cylindroid and associate cone are equal respectively to the half-sum and the half-difference of the perimeters of the bases is no longer true; and there is no necessity for making such restrictions. The theorem as stated above is universally applicable to solids of the ruled-surface type.

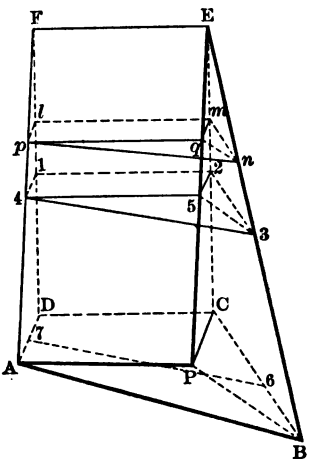

FIG. 10.

Let volume *FE-ABCD* be contained by a parallelogram *FECD*, two triangles *ADF* and *BCE*, a quadrilateral *ABCD*, and a warped surface *ABFE*. We have from Art. 10

$$V = H(\tfrac{1}{2}B - \tfrac{1}{6}A),$$

where B = area *ABCD*,

and A = area *CPB*.

Let a plane 1234 parallel to base *ABCD* bisect *FD*, *EC*, etc. Thus 1234 will be the mid-section ($S_{\frac{1}{2}}$) and 235 will be the mid-section of the associate pyramid, where *DCPA* is a parallelogram, and *CPB* is the base of the associate pyramid.

Now, area *DCPA* = 2 area 1254,

area *APB* = 2 area 354,

A = area *CPB* = 4 area 253 = $4T$.

Adding, we get

$$DCPA + APB + CPB = B = 2 \text{ area } 1254 + 2 \text{ area } 354 + 4T$$
$$= 2(\text{area } 1254 + \text{area } 354 + \text{area } 235) + 2T.$$

But $1254 + 453 + 253 = S_{\frac{1}{4}}$.

$$\therefore B = 2(S_{\frac{1}{4}} + T).$$

Substituting this value of B in the formula for the volume, we get

$$V = H(S_{\frac{1}{4}} + T - \tfrac{1}{6}A) = H(S_{\frac{1}{4}} + T - \tfrac{1}{6}T) = H(S_{\frac{1}{4}} + \tfrac{1}{6}T). \quad . \quad (2)$$

14. Project the area 1234 and the base in $CD76$, where 6 and 7 are the mid-points of CB and DA respectively. The perimeter of the mid-section is equal to the perimeter of $D76C = DC + C6 + D7 + 67$. All the terms of this equality can be obtained as halves of the sides of the bases, except the side 67. This cannot be obtained as a simple fractional part of DC and AB. If the line EF is parallel to AB, the point P falls on AB, and then if 6 and 7 are the mid-points of CB and AD (that is, if $ABFE$ is not warped), 67 (= 43) is the half-sum of AB and DC, and PB is the difference of EF and AB, and 53 will become the half-difference.

If $AD = PC = b$, and $CB = a$, angle $PCB = \theta$, we have

$$PB = \sqrt{(a - b)^2 + 4ab \sin^2 \frac{\theta}{2}}.$$

15. Let ABC-DEF, Fig. 11, be a solid terminated by two triangular bases ABC and DEF and whose mantle is composed of three warped surfaces $ABDF$, $BCDE$, and $ACEF$.

Project the upper base ABC on the plane of the lower base in abc. Draw EP and bP parallel to cb and cE respectively, EQ and aQ parallel to ac, and cE and aR and DR parallel to bD and ab. Join PD, FQ, and FR. Take any point S (Fig. 12) in plane of upper base ABC, and draw lines SN, SL, and SG parallel and equal to BD, CE, and AF of Fig. 11. S-GNL is the associate pyramid for the solid ABC-DEF. From the preceding article the triangles DPb, FQa, and FRa are the bases of the associate pyramids for the solids CB-$DEbc$, AC-$EFac$, and AB-$FDab$. Draw SM (Fig. 12) parallel to Bb (Fig. 11) and let it cut plane of triangle GNL in point M. Join MN, ML, and GM. It is clear that SM will be equal to Bb. Now, as SN and SM are parallel and equal to BD and Bb, the triangles SMN and BbD are equal. Hence $Db = MN$. The triangles SNL and BDP have the sides SN and SL parallel and equal to BD and BP. Therefore NL is parallel and equal to DP. The triangles MNL and DPb have the sides MN and NL parallel and equal to Db and DP

respectively. They are therefore equal and the sides ML and Pb are equal and parallel. But Pb is parallel and equal to Ec and Qa.

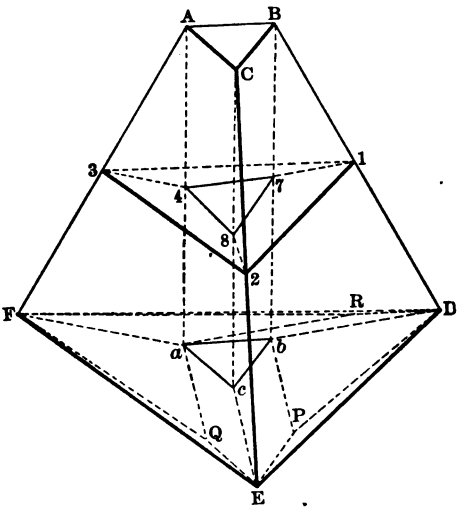

FIG. 11.

The triangle SGL has its sides SG and SL parallel and equal to AF and AQ of AFQ. $\therefore$ FQ is parallel and equal to GL. The triangles

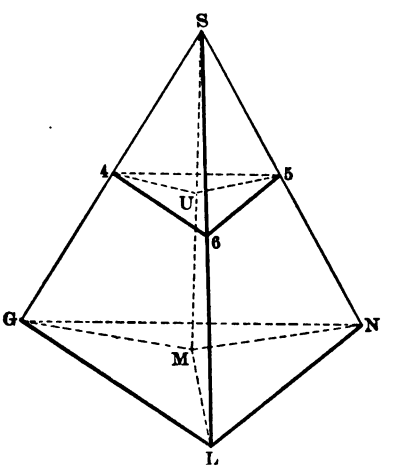

FIG. 12.

GML and FQa are therefore equal. In the same way we can show that triangle GMN is equal to the triangle FaR. Thus the base of

the associate pyramid *S-GLN* is equal to the sum of the bases of the associate pyramids of the individual solids composing the entire solid.

Let 123784 and 456*U* be the mid-sections of the solid and of the associate pyramid.

From the preceding proposition we have

$$\text{Vol. } CB\text{-}DEcb = H[1278 + \tfrac{1}{3}(U65)];$$

$$\text{Vol. } AC\text{-}FEac = H[2843 + \tfrac{1}{3}(U64)];$$

$$\text{Vol. } AB\text{-}FDba = H[1743 + \tfrac{1}{3}(U45)].$$

And we also know that

$$\text{Vol. prism } ABC\text{-}abc = H(ABC) = H(784).$$

Adding, we get

$$\begin{aligned}\text{Vol. } ABC\text{-}DFE &= H[123 + \tfrac{1}{3}(456)] \\ &= H[S_{\frac{1}{3}} + \tfrac{1}{3}T] \\ &= H[S_{\frac{1}{3}} + \tfrac{1}{12}A].\end{aligned}$$

16. From the prismatoid it is readily seen that the perimeter of the mid-section is the half-sum of the perimeters of the bases; and the perimeter of the base of the associate pyramid is the algebraic difference of the perimeters of the bases. Note that it is necessary to regard all contra-clockwise motion as negative, and all bases thus generated. The perimeter of the mid-section of the associate pyramid is therefore half the difference of the bases.

The above form of stating the theorem is adopted as being more compact and of more general application than the form in which Koppe stated it.

16a. We have already established the associate formula in Arts. 6 and 7. It can readily be established for Figs. 11 and 12.

From Art. 10 we have

$$\text{Vol. } CB\text{-}DEbc = H[\tfrac{1}{3}(EDbc) - \tfrac{1}{6}(DPb)];$$

$$\text{Vol. } AC\text{-}FEac = H[\tfrac{1}{3}(FEca) - \tfrac{1}{6}(FQa)];$$

$$\text{Vol. } AB\text{-}FDab = H[\tfrac{1}{3}(FDab) - \tfrac{1}{6}(FRa)];$$

$$\text{Vol. } ABC\text{-}abc = H(ABC).$$

Adding and replacing *DPb*, *FQa*, and *FRa* by their equivalents *LMN*, *GML*, and *GMN*, we have

$$\begin{aligned}V &= H[\tfrac{1}{3}(EDbc + FEca + FDab + ABC) \\ &\qquad + \tfrac{1}{3}ABC - \tfrac{1}{6}(LMN + GML + GMN)] \\ &= H[\tfrac{1}{2}(B_1 + B_2) - \tfrac{1}{6}A].\end{aligned}$$

17. *The mean area of prismoid (cylindroid) is equal to one-fourth of the sum of either base and three times a parallel section at two-thirds the altitude from said base.*

In Fig. 10 let section *lmnpq* be determined by a plane parallel to base *DCBA* and at the normal distance of two-thirds the altitude from the base.

From Art. 10 we have

$$V = H\left(\frac{B}{2} - \frac{A}{6}\right),$$

or $$M = \tfrac{1}{2}B - \tfrac{1}{6}A.$$

For convenience let

x = area of parallelogram *lmpq*;

y = area of triangle *pqn*;

z = area of triangle *mqn*;

$T = lmnp = x + y + z$.

Then we have

area $DCPA = 3x$;

area $APB = 3y$;

area $CPB = 9z = A$.

Adding, we get

$$B = 3x + 3y + 9z.$$

Substituting this value of B in the expression for (M), we get

$$M = \frac{3x + 3y + 9z}{2} - \frac{9z}{6}$$

$$= \frac{(3x + 3y + 9z) + 3(x + y + z)}{4}$$

$$= \tfrac{1}{4}(B + 3T). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Of course if *ABFE* is not warped, areas *ABP* and y disappear entirely. In Fig. 11 let T_3, T_4, T_5, and T_6 be the sections of the solids *CB-DEbc*, *AC-FEac*, *AB-FPab*, and *ABC-abc* at two-thirds the altitude from base *FDE*, and let the bases of these solids be B_3, B_4, B_5, and B_6.

Then we have

$$\text{Vol. } CB\text{-}DEbc = \frac{H}{4}(B_3 + 3T_3);$$

$$\text{Vol. } AC\text{-}FEac = \frac{H}{4}(B_4 + 3T_4);$$

$$\text{Vol. } AB\text{-}FDab = \frac{H}{4}(B_5 + 3T_5);$$

$$\text{Vol. } ABC\text{-}abc = \frac{H}{4}(B_6 + 3T_6).$$

Adding, we have

$$V = \frac{H}{4}\Big[B_3 + B_4 + B_5 + B_6 + 3(T_3 + T_4 + T_5 + T_6)\Big]$$

$$= \frac{H}{4}(B_1 + 3T).$$

18. This theorem can readily be established for the prismatoid. Divide the prismatoid into the three classes of tetrahedra as in Art. 6. Let *stu*, *lmno*, and *pqt* be the sections parallel to bases in the primary solids (*a*), (*b*), (*c*) of Fig. 5, at a distance of two-thirds the altitude from the lower base.

Now in Fig. 5 (*a*) we have

$$stu = \tfrac{4}{9}EFG = \tfrac{16}{9}d,$$

and in Fig. 5 (*b*)

$$lm = \tfrac{4}{9}bc,$$

$$mn = \tfrac{2}{9}dc.$$

$$\therefore lm \times mn = \tfrac{8}{9}bc \times dc.$$

Therefore the parallelogram $lmno = \frac{8}{9}k$.

In Fig. 5 (*c*)

$$pqr = \tfrac{1}{9}BCD = \tfrac{1}{9}e.$$

The corresponding sections of *E-ABD* and *GE-BA* are $\frac{1}{9}f$ and $\frac{8}{9}g$

$$\therefore S_{\frac{2}{3}} = \frac{16d}{9} + \tfrac{8}{9}k + \tfrac{8}{9}g + \tfrac{1}{9}e + \tfrac{1}{9}f.$$

$$\text{But volume} = \frac{H}{6}[8d + 8e + 8f + 4g + 4k]$$

$$= H\left[\frac{4e + 4f}{4} + \frac{3}{4}\left(\frac{16d}{9} + \frac{8k}{9} + \frac{8g}{9} + \frac{4k}{9} + \frac{4e}{9}\right)\right]$$

$$= H\left(\frac{B_1}{4} + \frac{3}{4}S_2^2\right) = \frac{H}{4}[B_1 + 3S_2^2]$$

$$= \frac{H}{4}(B_1 + 3T).$$

19. Analytical Proof.—Let the cross-section be a cubic function of its distance from some reference parallel section.

$$A_x = a + bx + cx^2 + dx^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$

Suppose the altitude (H) divided into n equal parts and planes passed parallel to these bases. There will be $n - 1$ sections other than the two bases. The body will be divided into n solids, each having an altitude of $H \div n$. Upon each section commencing with first base erect cylinders (prisms). If we multiply the bases by the altitudes and sum the results, we shall obtain a volume greater or less than the volume of the whole solid. Let $v =$ volume of any cylinder.

$$\Sigma(v) = \text{sum of volumes of cylinders.}$$

$$\therefore \Sigma(v) = \{a\}\frac{H}{n} + \left\{a + b\frac{H}{n} + c\frac{H^2}{n^2} + d\frac{H^3}{n^3}\right\}\frac{H}{n}$$

$$+\left(a + \frac{2bH}{n} + \frac{4H^2c}{n^2} + \frac{8h^3d}{n^3}\right)\frac{H}{n} + \left(a + \frac{3bH}{n} + \frac{9H^2c}{n^2} + \frac{27H^3d}{n^3}\right)\frac{H}{n}$$

$$+\left(a + \frac{4bH}{n} + \frac{16H^2c}{n^2} + \frac{64H^3d}{n^3}\right)\frac{H}{n} + \ldots$$

$$+\left(a + b(n-1)\frac{H}{n} + c(n-1)^2\frac{H^2}{n^2} + d(n-1)^3\frac{H^3}{n^3}\right)\frac{H}{n}$$

$$= \frac{H}{n}\left\{na + \frac{bH}{n}[1+2+3+4\ldots(n-1)] + \frac{cH^2}{n^2}[1+4+9+16+(n-1)^2]\right.$$

$$\left. + \frac{dH^3}{n^3}[1 + 8 + 27 + 64 \ldots (n-1)^3]\right.$$

But

$$1 + 2 + 3 + 4 + 5 + 6 \ldots n = \frac{n}{2}(n+1);$$

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 \ldots n^2 = \tfrac{1}{6}n(n+1)(2n+1);$$

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 \ldots n^3 = \frac{n^2}{4}(n+1)^2.$$

$$\therefore \Sigma(v) = \frac{H}{n}\left\{ na + \frac{bH}{2n}(n-1)(n-2) + \frac{cH^2}{6n^2}(n-1)(n)(2n+1) + \frac{dH^3}{4n^3}(n-1)^2n^2 \right\}$$

$$= H\left\{ a + \frac{bH}{2}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \frac{cH^2}{6}\left(1-\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{dH^3}{4n^3}\left(1-\frac{1}{n}\right)^2 \right.$$

If n becomes indefinitely great this expression for $\Sigma(v)$ approaches V, the volume of the solid, and when n = infinity, $\Sigma v = V$.

$$\therefore V = H\left(a + \frac{bH}{2} + \frac{cH^2}{3} + \frac{dH^3}{4}\right)$$

$$= \frac{H}{6}(6a + 3bH + 2cH^2 + \tfrac{3}{2}dH^3).$$

If in (A) we make x = 0, $\frac{1}{2}H$, and H, we obtain

$$B_1 = a, \quad S_{\frac{1}{2}} = a + \frac{bH}{2} + \frac{cH^2}{4} + \frac{dH^3}{8}, \text{ and } B_2 = a + bH + cH^2 + dH^3.$$

$$\therefore B_1 + 4S_{\frac{1}{2}} + B_2 = 6a + 3bH + 2cH^2 + \tfrac{3}{2}dH^3.$$

$$\therefore V = \frac{H}{6}(B_1 + 4S_{\frac{1}{2}} + B_2). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

20. Two-term Formulæ.—If $d = 0$ in (A), the area becomes a quadratic function of x.

In A make $x = \frac{1}{3}H$ and $\frac{2}{3}H$.

$$\therefore S_{\frac{2}{3}} = a + \tfrac{2}{3}bH + \tfrac{4}{9}cH^2 + \tfrac{8}{27}dH^3;$$

$$\therefore S_{\frac{1}{3}} = a + \tfrac{1}{3}bH + \tfrac{1}{9}cH^2 + \tfrac{1}{27}dH^3;$$

$$\therefore B_1 + 3B_{\frac{2}{3}} = 4a + 2bH + \tfrac{4}{3}cH^2 + \tfrac{8}{9}dH^3.$$

But
$$V = \frac{H}{4}(4a + 2bH + \tfrac{4}{3}cH^2 + dH^3)$$

$$= \frac{H}{4}(B_1 + 3S_{\frac{2}{3}} + \tfrac{1}{3}dH^3).$$

Then we see that if $d = 0$, we get a two-term formula. We also have

$$B_2 + 3S_{\frac{1}{2}} = 4a + 2bH + \tfrac{3}{4}cH^2.$$

$$\therefore V = \frac{H}{4}(B_2 + 3S_{\frac{1}{2}}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

That is, if we have a prismoid or one of the allied forms of solids, we can obtain the volume by using one base and a section at two-thirds of the altitude from said base. All cross-sections of earthwork where the road-bed is rising or falling gradually are quadratic functions of their distances from one base. One of the problems below gives a case where the cross-section is a function of the fourth degree of the running distance.

It can also be shown that if we take sections S_1 and S_2 at $\frac{H}{6}(3 - \sqrt{3})$ and at $\frac{H}{6}(3 + \sqrt{3})$ respectively, that is, at about 21′ and 79′, where $H = 100'$, we can obtain the volume by the following formula:

$$V = \frac{H}{2}(S_1 + S_2).$$

Make x in (A) equal to these values respectively.

$$\therefore S_1 = a + \frac{b}{6}(3 - \sqrt{3})H + \frac{cH^2}{6}(2 - \sqrt{3}) + \frac{dH^3}{36}(9 - 5\sqrt{3});$$

$$S_2 = a + \frac{b}{6}(3 + \sqrt{3})H + \frac{cH^2}{6}(2 + \sqrt{3}) + \frac{dH^3}{36}(9 + 5\sqrt{3}).$$

$$\therefore \tfrac{1}{2}(S_1 + S_2) = a + \frac{bH}{2} + \tfrac{3}{4}cH^2 + \frac{dH^3}{4}.$$

But $$V = H(a + \frac{bH}{2} + \tfrac{3}{4}cH^2 + \tfrac{1}{4}dH^3).$$

$$\therefore V = \frac{H}{2}(S_1 + S_2).$$

In an earthwork cut where the stations are 100′ apart these stations (S_1 and S_2) should be taken at 21′ and 79′ from one base or 21′ from each base. The error is less than two inches, the distances being 21′.14 and 78′.86.

21. Simpson's Rule.—Let 374 be part of a curve. If the points 3 and 4 are taken sufficiently near the curve, 374 can be regarded as a

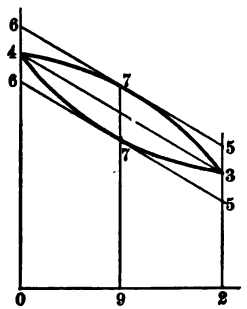

FIG. 13.

parabola. If 97 is the mid-ordinate and we join 34 and draw a tangent 56 at 7, it will be parallel to 43 and we have (347) = ⅔(3564).

Let $o4 = y_1$, $79 = y_{\frac{1}{2}}$, $23 = y_2$, $o2 = H$.

$$\text{Area } (o23740) = \text{area } (o234) \pm \text{area } (473)$$

$$= \frac{H}{2}(y_1 + y_2) \pm \tfrac{2}{3}(4356)$$

$$= \frac{H}{2}(y_1 + y_2) \pm \tfrac{2}{3}H[y_{\frac{1}{2}} \mp \tfrac{1}{2}(y_1 + y_2)]$$

$$= \frac{H}{6}(3y_1 + 3y_2 + 4y_{\frac{1}{2}} - 2y_1 - 2y_2)$$

$$= \frac{H}{6}(y_1 + 4y_{\frac{1}{2}} + y_2).$$

To find the area of a curvilinear figure: Divide the base (its projections or any line) into an even number of parts of width h. Let the first ordinate be y_1 and the final ordinate yn; then

$$A = \frac{2h}{6}(y_1 + 4y_2 + 2y_3 + 4y_4 + y_5 + 4y_6 \ldots y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{2h}{6}[(y_1 + y_n + 4(y_2 + y_4 \ldots ym_{n-1}) + 2(y_3 + y_5 \ldots y_{n-2})]. \quad \text{(B)}$$

That is, add together end ordinates, four times the even ordinates, and twice the other odd ordinates, and multiply the sum by one-third

of the base of each. This rule can be used for finding the volume of earthwork by inserting *volume* for area, and *areas* for ordinates.

If there is an even number of ordinates, find area of first three segments by two-term formula, and remainder by formula (B).

$$A = \frac{h}{2}(y_1 + 3y_2) + \frac{h}{3}[y_4 + y_n + 4(y_5 + y_7 + y_{n-1}) + 2(y_6 + y_8 \ldots y_{n})].$$

CHAPTER II.

PRISMOIDAL FORMULÆ APPLIED TO EARTHWORK.

22. Slope-stakes.—Stakes are always set in the cross-sections at the station where the side slopes intersect the terrain. Let the cross-section $ABCDC'B'$ intersect the side slopes and the ground surface

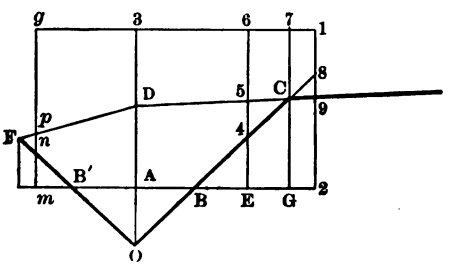

FIG. 14.

on the right in BC and DC. The point C will be the stake point on the right of the centre. We shall let c = centre cut DA, a = half width of road-bed, h = side height CG, r = rod reading on $D = D3$, and s = ratio of BG to $CG = \frac{BG}{CG}$ = tangent of BCG. If the heights of all points are referred to a horizontal plane through AB, $A3$ will be the height of the instrument (H) for the line of sight 367.

If we place the rod at any point whatever (as 4) on BC, its perpendicular distance from the centre line AD will be equal to d, where

$$d = AE = AB + BE$$
$$= a + s(4E) = a + s \text{ (height of pt. above datum).}$$

The problem in setting slope-stakes is to find the point C. Let H = height of instrument above datum = $6E = 7G$. Then if r' = rod reading at any distance out (d), we should have

$$d = a + s(H - r').$$

The point C can be found only by trial. Suppose we try a point (5) on the ground surface on right of station. The rod reading is 65;

the height of point above datum is $5E$. The calculated distance out (d) is therefore

$$d_c = a + s(5E) = a + s(4E) + s(54).$$

Now, the true distance out $= AE = AB + BE = a + s(4E)$.

$$\therefore d_c = d + s(54).$$

Hence we see that for 5 the calculated distance out is too great; or, in other words, when the point is within the cut (not sufficiently far out) the calculated distance is greater than the true distance from centre to point.

Try the point 9. There we would find our calculated distance out to be

$$d_c = a + s(92).$$

But true distance $d = a + s(82) = a + s(92) + s(89)$.

$$\therefore d_c = d - s(89), \quad \text{or} \quad d = d_c + s(89).$$

That is, if the calculated distance out is less than the measured distance out, the assumed point is *too far out.*

Try a point p on the surface to the left of station. We have

$$H = gm, \quad pg = r', \quad \text{height of } p = pm.$$

$$\therefore \text{height of point } p \text{ above datum} = H - pg = H - r'.$$

Cal'd. dist. out $d_c = a + s(H - r') = a + s(pm) = a + s(pn) + s(nm)$.

But true dist. out $d = a + s(nm)$. $\therefore d_c = d + s(pn)$.

That is, if the calculated distance out is *greater* than true (measured) distance out, the stake-point is further out than the assumed point.

22a. The Location of Slope-stakes in Fills.—Fig. 15 represents a fill where BB' is width of road-bed and $D9$ is the intersection of the vertical plane of cross-section with the ground surface. Try any point by taking rod readings. We already have $c = AD$, and by taking a rod reading on $D(D3)$, where the line of sight is 367, we get

$H = 3A = 6E$, $r = 3D$.
Or $H = r - c = A3 =$ height of instrument.
Read rod at any point 4. Rod reading $= 64 = r'$.
Height of $4 = E4 = 64 - E6 = r' - (r - c) = r' - H$.
But $d_e = a + s(E4) = a + s(E5) + s(45)$.
But true distance out $d = a + s(E5)$.

$$\therefore d_e = d + s(45).$$

We therefore conclude that if the calculated distance out is *greater* than the measured or true distance out, the stake-point is further out

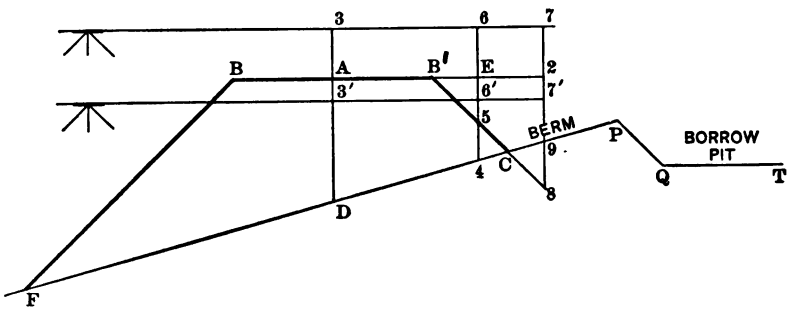

FIG. 15.

than the assumed, and *vice versa*. For every point on BC the actual and computed rod readings ought to agree. The actual rod readings at 4 and 9 are 64 and 79, while the computed readings will be 65 and 78.

If the line of sight is below road-bed (datum), as 3'6'7', we have rod reading at $D = D3'$, centre cut $= DA$.

$$\therefore H = D3' - DA = -A3' = -E6' = -27'.$$

Calculated distance out at any point $4 = a + s_1(E4)$, which is greater than the true distance out $[a + s(E'5)]$. At any point 9 without c the calculated distance out $= a + s(92) = d_c$.

But true distance out $= a + s(28) = a + s(29) + s(89) = d$.

$$\therefore d = d_c + s(89).$$

In conclusion we can formulate the following rule for excavation and embankment: If *the calculated distance* out is $\left(\begin{matrix}\text{greater}\\ \textit{less}\end{matrix}\right)$ than the *measured* distance, the stake point is further $\left(\begin{matrix}\text{out}\\ \text{in}\end{matrix}\right)$.

If we place the rod at 5 (Fig. 14) at a distance out a, we can readily find what the rod reading ought to be if 5 is a stake point. We have

$$d = AE = BE + a.$$

$$\therefore BE = d - a.$$

But $$BE = s(E4). \quad \therefore E4 = \frac{d-a}{s}.$$

The computed rod reading (r) would therefore be

$$r = 64 = H - E4 = H - \frac{d-a}{s} = 6E - E4.$$

That is, to find the computed rod reading subtract the half-width of road from the measured distance out; divide the difference by the

slope ratio, and subtract the quotient from the height of instrument. But the actual rod reading at 5 would be 65. Thus if the computed rod reading (r) is greater than the actual rod reading, the stake point is further out, and *vice versa*. This conclusion can readily be drawn from Fig. 14 without any calculation whatever. The computed rod reading (r) will always be the vertical distance between lines BC and 367, and the real rod reading will be the vertical distance between DC and 367. Then at all points *within* the stake point the computed rod reading is greater than the actual. When the assumed point is *without* C the computed rod reading (r_c) = 18 will always be less than the actual (19). In embankments (Fig. 15) if the assumed point 4 is within the stake point, the computed rod reading (65) is less than the actual and *vice versa*. This can be embodied in the following rules:

1. If the computed rod reading in excavations is $\left(\begin{matrix}\text{greater}\\ \text{less}\end{matrix}\right)$ than the true rod reading, the stake point is further $\left(\begin{matrix}\text{out}\\ \text{in}\end{matrix}\right)$.

2. If the computed rod reading in embankments is $\left(\begin{matrix}\text{greater}\\ \text{less}\end{matrix}\right)$ than the actual rod reading, the stake point is further $\left(\begin{matrix}\text{in}\\ \text{out}\end{matrix}\right)$.

23. All necessary formulæ for finding the volume of earthwork solids have been derived in Chapter I, and no attempt will be made to use any new formulæ in this chapter. Our methods shall be along the old and well-tried lines, as these methods are well established and are sufficient for our purpose. But it must be remarked that the term "the prismoidal formula" is misleading, as there is, as can be seen by referring to Chapter IV, an infinite number of formulæ more characteristic of the prismoid than the Newtonian formula.

The general term *prismatoid* includes all forms of earthwork whose side faces are triangles or trapezoids; and as the formulæ of Newton and Kinklin apply to the prismatoid and cylindroid, it is clear that they will apply with exactness to all forms of earthwork that fall in the class of ruled surfaces. Earthwork is generally classed according to the number of ordinates required to determine the areas of the cross-sections as level, two-level, three-level, five-level, and irregular. The burden of the operation of finding the volume of earthwork cuts or fills consists in obtaining the areas of the cross-sections, and it will be only necessary to develop methods by which these areas can be found correctly and with the minimum amount of labor.

All stake points are referred to the centre of road-bed as a reference point. The height of a point is generally recorded as the numerator of a fraction, while its distance out is the denominator. These two coordinates are sufficient for location of point.

24. Level Ground. — When the transverse slope of the cross-section is zero, all heights above a line in the plane of the cross-section being equal, the ground is classed as level although it may

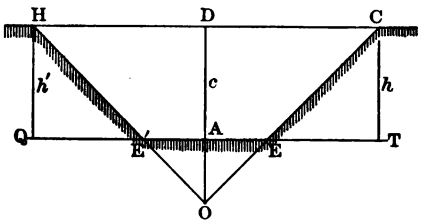

FIG. 16.

have any slope whatever longitudinally. The form of such a section is shown in Fig. 16, where DA = centre cut, EE' = width of road-bed, and s = tangent of angle ECT = tan AOE = ratio of slope.

Let $DA = c$.

$$CT = HQ = h = c.$$

Then

$$EE' = 2a\,;$$

$$ET = sh\,;$$

$$AT = a + sh = QA;$$

B = area of cross-section

$$\begin{aligned} &= HQTC - HQE' - CET \\ &= DA \times QT - \tfrac{1}{2}HQ.QE' - \tfrac{1}{2}CT.ET \\ &= c(2a + sh + sh') - \tfrac{1}{2}sh^2 - \tfrac{1}{2}sh'^2 \\ &= 2ac + sh^2 + sh'^2 - sh^2 \\ &= 2ac + sh^2 \\ &= 2ac + sc^2. \qquad \ldots\ldots\ldots \quad (5) \end{aligned}$$

If we find the values of B for different values of c, and locate the points corresponding to these values of c and B, we obtain the curve from which all areas can be read off for any special value of c. The values of c are laid off and a horizontal line and verticals are erected at the different points equal to the value of B, corresponding to that value of c. A curve (parabola) is then drawn through the points thus located.

Let $C = DO = DA + AO$.

In the right angle triangle AOE we have

$$\tan AOE = \frac{AE}{AO} = s\,;$$

$$AO = AE \div s = \frac{a}{s}\,;$$

$$C = c + \frac{a}{s}. \qquad \therefore c = C - \frac{a}{s}.$$

Substitute this value of c in equation (5), and there results

$$B = 2a\left(C - \frac{a}{s}\right) + s\left(C - \frac{a}{s}\right)^2$$
$$= sC^2 - \frac{a^2}{s}.$$

Now the area of the grade triangle $EOE' = \frac{1}{2}EE' \cdot AO$
$$= \tfrac{1}{2}2a \cdot \frac{a}{s} = \frac{a^2}{s}.$$

Therefore if we let $B' =$ cross-section + grade triangle, we have

$$B' = sC^2 - \frac{a^2}{s} + \frac{a^2}{s} = sC^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This, as the reader will readily recognize, is a parabola referred to its vertex as origin. If the diagram is constructed for equation (6) and a line is drawn parallel to C-axis at a height of $\frac{a^2}{s}$, the ordinates from this line to the curve will give the area of the cross-section. A complete diagram would include a curve for all the probable values of s.

25. Two-level Ground.—Where the ground has a uniform transverse slope, it is classed as two-level. Fig. 17 illustrates the form of the sections, where the straight line HC is the intersection of plane of cross-section with ground surface.

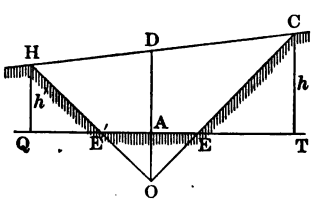

FIG. 17.

B = area of cross-section = $HQTC - HE'Q - CET$.

But area $HQCT = \frac{1}{2}(HQ + CT) \times QT$
$= \frac{1}{2}(h' + h)(EE' + E'Q + ET)$;

area $HQTC = \frac{1}{2}(h + h')(2a + sh + sh')$;

area $HE'Q = \frac{1}{2}HQ \cdot E'Q = \frac{1}{2}sh'^2$;

area $CET = \frac{1}{2}sh^2$.

$$\therefore B = \tfrac{1}{2}(h + h')(2a + sh + sh') - \tfrac{1}{2}sh^2 - \tfrac{1}{2}sh'^2$$
$$= shh' + a(h + h'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

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In this formula for area of cross-section the half-width (a) and slope ratio (s) are constant for any special piece of work. Diagrams have been constructed by the late A. M. Wellington from which the area is read off for any special values of h and h'. These diagrams do the multiplying for us instantaneously, and have one decided advantage over the use of the formula, and that is that no sensible error can occur due to multiplication. If we take two axes at right angles, and let the abscissas represent h and the ordinates represent B, we can find the value of h', as s and a are known. But as there is an infinite number of values of h and B that will give the same value of h', it is clear from (7) that for any special value of h' the equation will represent a straight line.

Example.—Let $2a = 20$; $s = \frac{2}{3}$.

$$\therefore B = \tfrac{2}{3}hh' + 10(h + h').$$

Assume $h' = 10$. $\therefore B = 15h + 10h + 100 = 25h + 100$.

Now if we give to h different values and find the corresponding values of B, and lay off the values of h along OX from O, and at the points of division erect verticals equal to the corresponding values

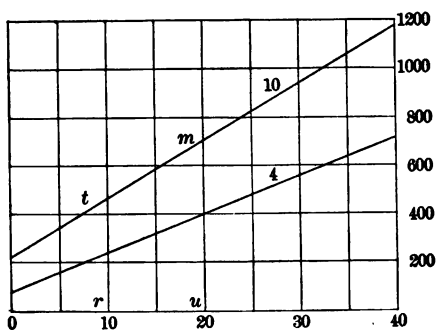

FIG. 18.

of B, we locate points on a line called the h'-line. In the present example, where $h' = 10$, the line is called the 10-line, etc.

If $h = 10$, $B = 350$; and when $h = 20$, $B = 600$.

Lay off $Or = 10$, and make rt to a different scale $= 350$.

Make $ou = 20$, and erect $um = 600$. Join tm. This is now called the 10-line, and by making h equal to any value whatever, and $h' = 10$, we can read off the value of B. Thus if $h' = 15$, we get $B = 475$; if $h' = 5$, we get 225 for B; if $h' = 0$, $B = 100$. To locate the 4-line, we have

$$B = 6h + 10h + 40 = 16h + 40.$$

If $h' = 10$, $B = 200$; and if $h' = 20$, $B = 360$. From these values of the coordinates we can construct the 4-line.

In the same way we can construct all the sloping lines for this width of road-bed (20) and ratio of slope 3 : 2. When the diagram is completed to large scale for all values of h' from 0 to 50, it offers the quickest and most direct way of finding any probable areas for two-level sections.

If $2a = 18$ and $s = \frac{3}{2}$, we have

$$B = \tfrac{3}{2}hh' + 9(h + h').$$

Let $h' = 4$ and make $h = 7$ and 17. The respective values of B are 130 and 270. By making h' equal to any other value we can locate the line for that value of h'. Thus we can construct another diagram for the width 18 and slope 3 : 2.

If the slope is 1 : 1, these formulæ become

$$B = hh' + 10(h + h');$$

$$B = hh' + 9(h + h').$$

Diagrams can be constructed for any combination of road-bed and slope whatever; and where the excavations are made in soils whose angles of repose or whose slopes vary, the number of diagrams would become burdensome. Attention is here directed to Diagram 1, explained under the head of three-level ground. Its applicability to all three-level sections renders the work easy and reduces the number of diagrams from a dozen (say) to one.

26. If we make B constant in formula (7), the formula assumes a convenient mathematical form, but one that can be graphically used without tedious interpolations, as the vertical spacings between the different curves corresponding to different values of B is not constant.

Now in Fig. 17 we have $CT = h$, and $HQ = h'$. Let x and y represent the heights of the points C and H above O. We then have

$$x = h + AO = h + \frac{a}{s},$$

and

$$y = h' + \frac{a}{s};$$

or

$$h = x - \frac{a}{s},$$

$$h' = y - \frac{a}{s}.$$

By substituting these values of h and h' in (7) and reducing, we get

$$sxy = B + \frac{a^2}{s} = \text{area } HOC. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7')$$

If the area HOC is made constant (say 200), and values are given to x and y, and the points located as indicated above, the locus of these points will be a curve that should be marked "200." In the same way other curves can be found for values of B, i.e., say 100, 200, 400, etc. The curves thus constructed are equilateral hyperbolas, and the objection obtains to their use that was mentioned above. However, if we make y constant and construct lines for x and B as coordinates, we get a series of straight lines as in Fig. 18, except that they all pass through O, and in using them for finding the area we must bear in mind that they give the area HOC which includes the grade triangle. This, of course, mast be subtracted before the area can be used in the formula for finding the volume.

If the ground slopes transversely at r, that is, makes an angle $\tan^{-1} r$ with the horizontal, and if h = side height on upper side, h' = side height on lower side, etc., the following relations are easily deduced:

$$h = (c + ra) \div (1 - rs);$$
$$h' = (c - ra) \div (1 + rs);$$
$$B = sc^2 + 2ca + a^2rs. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (X)$$

If the surface ground is rising or falling at a uniform rate longitudinally, c becomes a function of z (the distance from any one station). If now the road-bed is horizontal and the surface ground falls p ft. in 100, we have

$$c_z = c - \frac{pz}{100}.$$

If we substitute this value of c_z in (X), we get the general expression for the running section. This, under the conditions, is a quadratic function of z.

27. *Where the algebraic difference of gradients is greater than .3 per cent, vertical curves are inserted at crests and sags in railroad work. The common parabola on account of its simplicity is usually employed for such purposes. Thus DC is the surface of the ground and AB the vertical curve (common parabola) whose vertex is at A and which is tangent to horizontal AE at A. If this falls away from horizontal AE an amount $EB = t$ in 100', then at any distance z from A it will fall an amount $tz^2 \div (10000)$.

If the ground surface DC falls longitudinally at a uniform rate, then we have

$$c_z = c_1 - \frac{pz}{100} + \frac{tz^2}{10000}.$$

* Nagle's Field Manual.

The cross-section (B) thus becomes a function of the fourth degree of the distance s, and *no "prismoidal" formula* will accurately give the volume of such a solid.

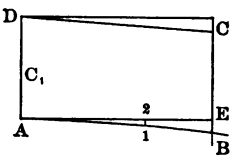

FIG. 19.

PROBLEM 1. The surface slopes transversely at 1 to 6 and is falling longitudinally at 6 to 100. If $s = \frac{2}{3}$, $2a = 18'$, and $c_1 = 24'$, find volume for length of 100′ if road-bed is horizontal.

PROBLEM 2. In the preceding problem if the centre line of road is graded to form of common parabola and its fall in 100′ amounts to 1′, find volume between A and B in Fig. 19.

28. **Three-level Sections.**—The central cut and side heights and distances out are always given in the field-notes. Sections that are

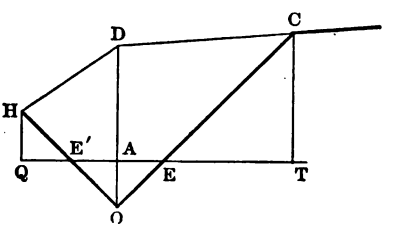

FIG. 20.

estimated from such data are called three-level sections. The central cuts are given from the level and grade lines, and the distances out are obtained when the slope-stakes are set.

As before we have the following:

Side height $CT = h$;

Side height $HQ = h'$.

Centre cut $AD = c$.

Width of road-bed $EE' = 2a$.

Distance out $AT = a + sh = d$;

Distance out $AQ = a + sh' = d'$.

Area $ECDHE'$ = area $ATCD$ + area $ADHQ$ − area ETC − area $E'HQ$

$$= \tfrac{1}{2}(c + h)(a + sh) + \tfrac{1}{2}(c + h')(a + sh') - \tfrac{1}{2}sh^2 - \tfrac{1}{2}sh'^2.$$

$$\therefore 2B = 2ac + (h + h')(a + sc) = c(d + d') + a(h + h').$$

But $$d + d' = 2a + s(h + h');$$

$$\therefore h + h' = (d + d' - 2a) \div s.$$

$$\therefore 2B = c(d + d') + \frac{a}{s}(d + d - 2a).$$

$$B = \frac{1}{2}\left(c + \frac{a}{s}\right)(d + d') - \frac{a^2}{s}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Now $a^2 \div s$ = area of grade triangle EOE'.

Hence $$B + \frac{a^2}{s} = \text{area of } OCDH$$

$$= \frac{1}{2}\left(c + \frac{a}{s}\right)(d + d').$$

In Fig. 20 AO, the altitude of grade triangle, $= \frac{a}{s}$, and $c = DA$.

$$DO = c + \frac{a}{s} = \text{total centre height} = C.$$

But $$d + d' = QT' = \text{total width} = w;$$

$$\therefore \text{total area} = \tfrac{1}{2} \text{ centre height} \times \text{width},$$

or $$B = \tfrac{1}{2}Cw. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

But $$V = \tfrac{1}{6}(B_1 + 4S_{\frac{1}{2}} + B_2) = \frac{l}{6}B_1 + \frac{l}{6}4S_{\frac{1}{2}} + \frac{l}{6}B_2.$$

Or the volume is equal to the sum of three prisms whose bases are the bases of the prismoid and four times the mid-section, and whose altitude is the one-sixth part of the altitude of the prismoid.

If we make $l = 100'$, and reduce the volume to cu. yds., we have

$$V = \frac{100}{6 \times 27}(B_1 + 4S_{\frac{1}{2}} + B_2)$$

$$= \frac{100}{162}B_1 + (4S_{\frac{1}{2}})\frac{100}{162} + \frac{100B_2}{162}.$$

If, therefore, we multiply each end section and four times the mid-section by $\frac{100}{162}$ and add the products, we obtain the true volume in cu. yds.

The general formula for the volume-area will be

$$V = \frac{100}{324}\left(c + \frac{a}{s}\right)(d + d') = \frac{100}{324}CW. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Then to obtain the volume of the railroad prismoid we shall have to subtract the volume of the prism whose base is the grade triangle and whose altitude is 100'. This will have to be subtracted from each volume obtained. It is sufficient and gives less labor to add the volumes of the end sections and four times the mid-section and then subtract the volume of grade prism.

As the majority of cross-sections of railroad earthworks are estimated from either two or three levels, it will be well to develop some plan that will facilitate the calculations. Two plans of making these calculations have been used. One plan is the use of diagrams, and the other the use of tables.

If in equation (10) we make C constant, then V will vary directly as w and the equation will represent a straight line. If, for example, we make $C = 16'$, this line will be called the 16-line. Now lay off on

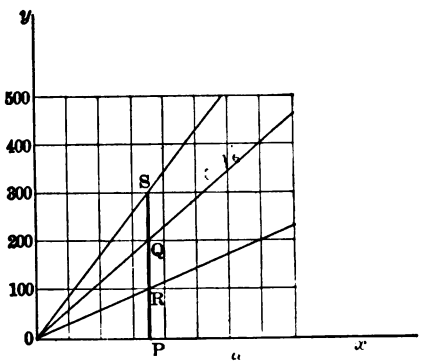

FIG. 21.

a horizontal ox from O the different widths, and at these points erect a line which to a convenient scale represents the volume. If in (10) we make $C = 18$, we have

$$V = \frac{100}{324} 18w = \frac{100}{18} w.$$

If we make $w = 0$, then $v = 0$; therefore the line passes through O. as will all other total centre-height lines. To plot this line, give w any value and lay this off on $ox = OP$. Let $w = 36 = OP$.

$$V = \frac{100 \times 36}{18} = 200 \text{ cu. yds.}$$

Mark off on OY equal divisions to represent 100 cu. yds. Through 200 draw a line parallel to OX to cut a line through P parallel to

OY. These lines intersect in a point Q. Join OQ and produce it. This line is called the 18-line and is so marked. If in (10) we make $w = 36$, we have

$$V = \frac{100}{324}C \times 36 = \frac{100C}{9}.$$

Make $C = 27'$; $\therefore V = 300$ cu. yds.

Make $C = 9'$; $\therefore V = 100$ cu. yds.

Now as $OP = 36'$, draw lines through 300 and 100 intersecting PQ in R and S. The line OR will be the 9-line and OS the 27-line.

If we have given $w = 54$ and $c = 18$, we find 54 on ox and follow it up to line 18, and then follow the horizontal from this point to OY and find the volume to be 300.

Diagram I (called Earthwork Diagram) is constructed similar to Fig. 21 to a larger scale. This diagram reads off the volumes that would be given by the prismoidal formula and by average end areas. A few examples will be given to illustrate its use.

APPLICATIONS.—*First Example.*—Given $a = 18$, $s = \frac{1}{2}$, $C = 24$, and $w = 63$. From Earthwork Diagram we look for 63 on the horizontal line and follow it up to the 24-line, and then follow the horizontal line to the right and read off the volume = 467 cu. yds. On same Plate we take from the diagram for "Volumes of Grade Prism," for width = 18 and $\frac{1}{2}$ curve, the volume of grade prism = 54 cu. yds.

Second Example.—The level notes at station (1) are ($\frac{14}{10}$, 16, $\frac{18}{16}$), and those at station (2) are ($\frac{19}{24}$, 12, $\frac{14}{8}$), and $2a = 18$, $s = \frac{1}{2}$.

For $w = 66$ and $C = 22$ we get $v = 448$
For $w = 60$ and $C = 20$ we get $4v = 1480$
For $w = 54$ and $C = 18$ we get $v = 300$

2228

Deduct volume of grade prism 200

Total volume.....2028 cu. yds.

If we wish to find the volume by averaging the mean end areas, that is, if we assume that $V = \frac{1}{2}(B_1 + B_2)$, we have, when the volume is estimated in cubic yards and $l = 100'$,

$$V = \frac{100}{54}B_1 + \frac{100}{54}B_2.$$

But
$$B = \frac{1}{2}\left(c + \frac{a}{s}\right)(d + d').$$

Hence the volume in cu. yds. of the prism whose base is the cross-section plus the grade triangle is represented as before:

$$V = \frac{100}{108}CW. \quad \ldots\ldots\ldots \quad (10')$$

Thus the volume here read is three times the partial volume that we would read for the same centre heights and widths by the prismoidal formula. The Earthwork Diagram can be readily used for the method of average end areas (A. E. A.) by reading off the volume corresponding to any values of C and W and by multiplying this result by three, or by reading off the results (volumes) from the left margin of Diagram I.

Example.—Given $c_1 = 12$, $c_2 = 16$, $h_1 = 10$, $h_2 = 14$, $d_1 = 24$, $d_2 = 30$, $2a = 18$, $h_1' = 18$, $h_2' = 20$, $d_1' = 36$, $d_2' = 39$, to find the volume from the diagram both by the aid of the prismoidal formula and by the average end-area formula. We have at station (1) $C = 18$ and $w = 60$, and at station (2) we have $C = 22$ and $w = 69$, and at the mid-section $C = 20$ and $w = 64\frac{1}{2}$.

By the prismoidal formula:

For $w = 69$ and $C = 22$, the volume..... =	468
For $w = 64\frac{1}{2}$ and $C = 20$, 4 times the volume =	1592
For $w = 60$ and $C = 18$ volume..... =	333
Gross volume.....	2393
Deduct one grade prism.....	200
Total volume.....	2193 cu. yds.

By average end areas:

For $w = 69$ and $C = 22$, the volume..... =	1404
For $w = 60$ and $C = 18$, the volume..... =	999
Gross volume.....	2403
Deduct one grade prism.....	200
Total volume.....	2203 cu. yds.

30. The correct volume of a prismoid is given by the prismoidal formula, or

$$V = \tfrac{1}{6}(B_1 + 4S_{\frac{1}{2}} + B_2).$$

But $$B_1 = \tfrac{1}{2}lC_1W_1, \quad B_2 = \tfrac{1}{2}lC_2W_2;$$

$$4S_{\frac{1}{2}} = (C_1 + C_2)(W_1 + W_2);$$

$$V = \tfrac{1}{12}(2C_1W_1 + 2C_2W_2 + C_1W_2 + C_2W_1).$$

Let $V_e = \dfrac{l}{2}(B_1 + B_2)$ = volume obtained by using average end-area formula.

Now $$V_e = \frac{l}{4}(C_1W_1 + C_2W_2) = \frac{l}{12}(3C_1W_1 + 3C_2W_2),$$

$$V_e - V = \frac{l}{12}(C_1W_1 + C_2W_2 - C_1W_2 - C_2W_1).$$

$$\therefore \text{error} = \frac{l}{12}(C_1 - C_2)(W_1 - W_2).$$

If the upper surface is plane this correction is positive, but if it is warped it can become negative. Generally this correction is positive, that is, the average end-area formula gives results *too large*. If we wish to reduce this correction to cu. yds. for lengths of 100′, we have

$$\text{Correction in cu. yds.} = \frac{100}{12 \times 27}(C_1 - C_2)(w_1 - w_2). \quad . \quad (11)$$

The correction entirely disappears if the total widths or the total centre cuts are equal. This correction is always positive if the larger total centre height corresponds to the larger total width; otherwise it is negative. For the example in Art. 30 $C = 11$ cu. yds.

31. By comparing formula (10) with (11) we see that there is a striking similarity where $(C_1 - C_2)$ represents C and $(w_1 - w_2)$ represents w, and we can find the correction for mean end areas from Diagram I. On the horizontal line look for $(w_1 - w_2)$ and follow the vertical line through this point to an inclined line that has the value of $(C_1 - C_2)$ on it, and read off the volume to the *right*. This volume is the correction.

Example.—Given the following notes, where $s = \frac{3}{4}$ and $2a = 18$: at station 1, $(\frac{1}{18}, 20. \frac{22}{24})$, and at station 2, $(\frac{1}{17}, 14, \frac{1}{18})$.

The volume of the grade prism = 200 cu. yds.

From Diagram I the volume by prismoidal formula is equal to 602 + 1916 + 370 = 2888 cu. yds.

The volume by average end areas = 1806 + 1110 = 2916.

In finding the "correction" if necessary to bring $(C_1 - C_2)$ and $(w_1 - w_2)$ within the practical limits of the Earthwork Diagram, multiply each by 10 and point off two decimal places in the final result. Thus we look for $(C_1 - C_2) = 6$, and $(w_1 - w_2) = 15$. It is really immaterial whether we take $(w_1 - w_2)$ along the horizontal line or for $(C_1 - C_2)$ along this line. In this example we shall multiply the value of $(C_1 - C_2)$ by 10 and divide the final result by 10. Follow the vertical through 60 on the horizontal line to the 15 line and read to the *right* and we get 277. Divide by our factor 10 and we get 27.7. The volumes by actual calculation are:

By the prismoidal formula = 2888 − 200 = 2688 cu. yds.

By average end areas = 2916.

Volume of associate pyramid = 28 cu. yds.

PROBLEM 3. Given $2a = 18$, $s = \frac{3}{4}$, and the following level-notes: at station 1, $(\frac{1}{18}, 20, \frac{22}{24})$, and at station 2, $(\frac{1}{17}, 18, \frac{1}{18})$. Find volume by use of Earthwork Diagram, by the prismoidal formula, and by the average end areas, and check by finding correction.

PROBLEM 4. Given $2a = 20$, $s = 1$, and the following field-notes: at station 1, $(\frac{1}{18}, 18, \frac{20}{20})$, and at station 2, $(\frac{1}{14}, 26, \frac{1}{18})$. Find volume from diagram by both methods, and check carefully by correction.

PROBLEM 5. Draw the cross-sections in the preceding problem, and explain why the correction is negative.

32. Where one side of road is in cut and the other side in fill, the volume obtained by subtracting the volume of the grade prism from $\frac{100}{162}Cw$ will give a result too small by the volume of the prism, whose base is $HE't$ in Fig. 22, and whose height is 100′. We have $OD = C$,

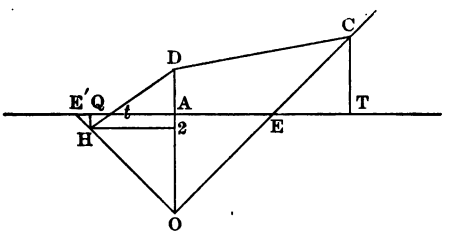

FIG. 22.

$AT = d$, $H2 = d'$, $CT = h$, and $A2 = h'$, where h' is negative. Area $OCDH = \frac{1}{2}C(d + d') = \frac{1}{2}CW$. Or, if modified for calculation by the prismoidal formula, we have

$$\text{Vol. } OCDH = \frac{100}{324}CW.$$

Now $OCDH = EOE' + tDCT - E'tH$.

But the volume excavated is obtained from the area $tECD$, and if we follow the usual rule and subtract the grade prism a result too small by the amount $\frac{l}{6 \times 27} \times (EtH)$ is obtained.

Now $H2 = a - sh' =$ dist. out of $H = d'$.

$E'Q = sh'$. Let $c = AD$. Then in the similar triangles $H2D$ and tAD we have

$$tA : H2 :: AD : D;$$

or
$$tA : a - sh' :: c : c + h', \quad \therefore tA = \frac{c(a - sh')}{c + h'}.$$

$$tE' = E'A - tA = a - \frac{c(a - sh')}{c + h'} = \frac{h'(a + sc)}{c + h'}.$$

$$\text{Area } E'tH = \tfrac{1}{2}QH \times E't = \tfrac{1}{2}h'^2\frac{2(a + sc)}{c + h'}.$$

33. Limitation of Coordinates.—While theoretically the coordinates C and w can have any values, there are certain limits in practice.

The width of road-bed ($2a$) is rarely less in railroad work than 14 ft., and it is not necessary to provide facilities for calculating volumes where widths of less than 14 ft. enter as factors.

Again, the width of cross-section in ordinary work is partially a function of the centre height. It is an exceptional case where this is not true. In nearly all cases when C is large w will be large, and it will be unnecessary to associate small values of C with large values of w. Of course, provision must always be made for the width of road-bed and for the altitude of the grade triangle. The maximum value of slope ratio (s) is about $1\frac{1}{2}$; the minimum value in earthwork (in practical usage) is about 1 : 8. If the width of road has its minimum value of 14 ft. the least altitude of grade triangle will be $7 \div 1\frac{1}{2} = 4.7'$, or to be safe say 4'. As this would admit centre heights of only 1', 2', and 3', and would save very little space either in diagrams or tables, these centre heights will be provided for.

34. To find volume when end sections are three-level and intermediates are five-level, i.e., when the upper surface is prismatoidal.

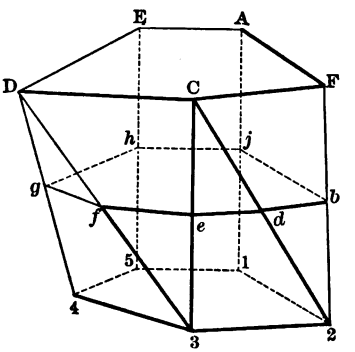

FIG. 23.

Let 12345 be the lower and $AFCDE$ the upper base of a railroad cut, where $15 = EA =$ width of road-bed $= 2a$, and $1A = E5 =$ length of cut or altitude.

Let centre cut at 3 $= c_1$;
Let centre cut at C $= c_2$;
Let side height at 2 $= h_1$;
Let side height at 4 $= h_1'$;
Let side height at $F = h_2$;
Let side height at $D = h_2'$.

Let the upper surface be determined by the four planes $FC2$, $C23$, $DC3$, and $D43$. The volume can be found by the formula of Newton

or any of the two-term formulæ. If B_1 and B_2 are the areas of the upper and lower bases respectively, we have

$$B_1 = 2ac_1 + (h_1 + h_1')(a + sc_1);$$
$$B_2 = 2ac_2 + (h_2 + h_2')(a + sc_2).$$

It is necessary to find the mid-section or section (say) at one third of the altitude from B_1 or B_2.

The mid-section (Fig. 23) *bdefghj* ($S_{\frac{1}{2}}$) can readily be found by projecting B_1 and $S_{\frac{1}{2}}$ on B_2 as in Fig. 24.

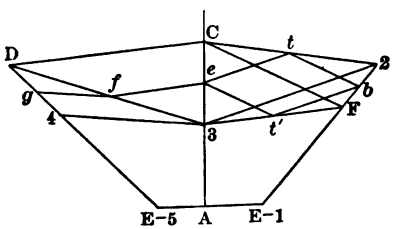

FIG. 24.

The following values of heights and distances out are easily verified from the figure:

Side height of point *b*	$= \frac{1}{2}(h_1 + h_2)$;
Height of *d*	$= \frac{1}{2}(c_2 + h_1)$;
Dist. out of *b*	$= \frac{1}{2}(d_1 + d_2)$;
Dist. out of *d*	$= \frac{1}{2}d_1$;
Height of *e*	$= \frac{1}{2}(c_1 + c_2)$.

$$S_{\frac{1}{2}} = \text{area } AetbE = \tfrac{1}{8}(2h_1 + h_2 + c_2)d_2 + \tfrac{1}{8}(c_1 + 2c_2 + h_1)d_1 - \tfrac{1}{8}s(h_1 + h_2)^2.$$

$$\therefore 4S_{\frac{1}{2}} = \tfrac{1}{2}(c_2d_2 + c_1d_1 + c_2d_1 + 3ah_1 + ah_2);$$

$$\therefore M = \tfrac{1}{6}(B_1 + 4S_{\frac{1}{2}} + B_2)$$

$$= \frac{1}{6}\left[\frac{c_1d_1}{2} + \frac{ah_1}{2} + \frac{c_2d_2}{2} + \frac{c_1d_1}{2} + \frac{c_2d_1}{2} + \frac{3ah_1}{2} + \frac{ah_2}{2} + \frac{c_2d_2}{2} + ah_2\right]$$

$$= \tfrac{1}{6}(c_1d_1 + c_2d_2 + c_2d_1 + 2ah_1 + ah_2)$$

$$= \tfrac{1}{6}[c_1d_1 + c_2d_2 + c_2d_1 + ah_1 + a(h_1 + h_2)]. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

34a. Formula (12) can be established independently of any previously derived formula. Let $KS7A$ be the half road-bed, where KS

is centre line. The prismoid on right of central plane (*CKS*3) will be *CKAF-S*327. Join point 2 to *A*, *K*, and *S*. The prismoid is com-

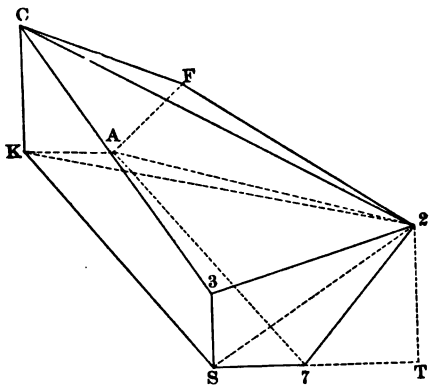

FIG. 25.

posed of pyramids 2-*AFCK*, 2-*KS*7*A*, and 2-*KS*3*C*. We therefore have

$$\text{Vol. } 2\text{-}AFCK = \tfrac{1}{3} \text{ altitude} \times \text{base} = \tfrac{1}{3}(KS) \text{ area } AFCK$$

$$= \frac{l}{6}(c_2 d_2 + a h_2) l;$$

$$\text{Vol. } 2\text{-}A7SK = \tfrac{1}{3} 2ah;$$

$$\text{Vol. } 2\text{-}KS3C = \tfrac{1}{3} ST \text{ area } KS3C = \frac{d_1}{6}(c_1 + c_2) l.$$

$$\therefore \text{ total volume} = \frac{l}{6}[c_1 d_1 + c_2 d_2 + c_2 d_1 + a h_1 + a(h_1 + h_2)].$$

This proof is taken essentially from Henck's Field-book.

35. If the upper surface of the ground on the right of the centre line is terminated by the planes 3*CF* and 3*F*2, a similar expression will be found for the mean area of the section. In this case the mid-section on the right of centre line will be *Aet'bE* (Fig. 24).

Height of $t' = \tfrac{1}{2}(c_1 + h_2)$;

Dist. out of $t' = \tfrac{1}{2} d_2$;

$$S_{\frac{1}{2}} = Aet'bE = \tfrac{1}{3}(h_1 + 2h_2 + c_1) d_1 + \tfrac{1}{3}(2c_1 + c_2 + h_2) d_2 - \tfrac{1}{3} s (h_1 + h_2)^2$$

$$= \tfrac{1}{3}(c_1 d_1 + c_2 d_2 + 2c_1 d_2 + 3a h_2 + a h_1).$$

But $M = \frac{1}{6}(B_1 + 4S_{\frac{1}{2}} + B_2)$

$$= \frac{1}{6}\left(\frac{c_1d_1}{2} + \frac{ah_1}{2} + \frac{c_1d_1}{2} + \frac{c_2d_2}{2} + c_1d_2 + \frac{3ah_2}{2} + \frac{ah_1}{2} + \frac{c_2d_2}{2} + \frac{ah_2}{2}\right)$$

$$= \tfrac{1}{6}[c_1d_1 + c_2d_2 + c_1d_2 + 2ah_2 + ah_1]$$

$$= \tfrac{1}{6}[c_1d_1 + c_2d_2 + c_1d_2 + ah_2 + a(h_1 + h_2)]. \quad . \ . \ . \quad (12')$$

Formulæ (12) and (12′) are of the same form and can be generalized into a

Rule.—To find the mean area when the upper surface is terminated by two triangles as in Fig. 23, add together the products of the centre cuts and distances out at each base, the product of the centre cut and distance out at the ends of the given diagonal, the product of width of half road-bed and side height that the diagonal joins, and the product of the sum of side heights and width of half road-bed ; take one-sixth of sum.

The mean area on left of centre for diagonal joining C and h' can be found by similarity :

$$M = \tfrac{1}{6}[c_1d_1' + c_2d_2' + c_1d_2' + ah_2' + a(h_1' + h_2')].$$

Thus the total mean area for diagonals $C2$ and $D3$ is found by adding (12) and (12).

$$\therefore M = \tfrac{1}{6}[c_1(d_1 + d_1') + c_2(d_2 + d_2') + CD + C'D' + a(h_1 + h_2' + h_1' + h_2 + H + {}_2H'],^* \quad . \ . \ . \ . \ . \ . \quad (13)$$

where C = centre cut to which diagonal on right is joined;
D = dist. out to which diagonal on right is joined;
C' = centre cut to which diagonal on left is joined;
D' = dist. out to which diagonal on left is joined;
H = side height to which diagonal on right is joined;
H' = side height to which diagonal on left is joined.

Example.—From the following notes compute the volume with diagonals in the different ways :

Sta.	L. C.	C. C.	R. C.
0	18 / 36	12	10 / 24
1	20 / 39	16	14 / 30

There are four different problems in this, according to the different combinations. The diagonal may run as indicated by the solid or

* See Henck's Field-book.

dotted lines. We shall find the mean area on each side of centre line separately.

(1) Mean area on right of centre with solid diagonal:

$$M = \tfrac{1}{6}[c_1d_1 + c_2d_2 + CD + aH + a(h_1 + h_2)]$$
$$= \tfrac{1}{6}[12 \times 24 + 16 \times 30 + 12 \times 30 + 9 \times 14 + 9(10 + 14)]$$
$$= \tfrac{1}{6}[288 + 480 + 360 + 126 + 216] = \tfrac{1}{6} \times 1470 = 245 \text{ sq. ft.}$$

$$\therefore \text{volume} = 24500 \div 27 = 907.41 \text{ cu. yds.}$$

(2) Mean area on right of centre, dotted diagonal:

$$M = \tfrac{1}{6}[12\times24+16\times30+16\times24+9\times10+9(10+14)] = \frac{1458}{6} = 243 \text{ sq. ft.}$$

$$\therefore \text{volume} = \frac{243 \times 100}{27} = 900 \text{ cu. yds.}$$

(3) Mean area on left of centre line when surface is divided by solid diagonal:

$$M = \tfrac{1}{6}[12\times36+16\times39+16\times36+9\times18+9(18+20)] = \frac{2136}{6} = 356 \text{ sq. ft.}$$

$$\therefore \text{volume} = \frac{356 \times 100}{27} = 1318.52 \text{ cu. yds.}$$

(4) Mean area on left of centre line when surface is divided by dotted diagonal:

$$M = \tfrac{1}{6}[12 \times 36 + 16 \times 39 + 9 \times 20 + 9(20 + 18\ ] = 341 \text{ sq. ft.}$$

$$\therefore \text{volume} = \frac{341 \times 100}{27} = 1262.96 \text{ cu. yds.}$$

36. To find the volume when the upper surface is warped:

Conceive the upper surface of the earthwork cut (Fig. 26) to be generated by a straight line which moves from position DC to position 43, so that the end points of the variable sect move with a uniform velocity. This moving line thus generates a warped or *ruled* surface. The same surface $DC34$ may be generated by a line moving from position $4D$ to position $3C$ subject to the same conditions, that is, that the end points of the moving sects travel at a uniform velocity. The volume generated by $PQ76$ or the volume contained by the planes $ABCD$, $AB21$, $BC23$, $AD41$, 2341. and the

warped surface $DC34$ is an arithmetical mean between the volumes determined by considering the upper surface composed of planes

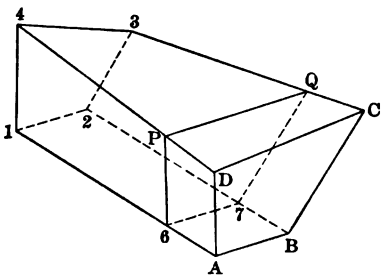

FIG. 26.

determined by the two diagonals $D3$ and $4C$. If the surface is divided by the diagonal $C4$, the vertical plane $P67$ will cut the

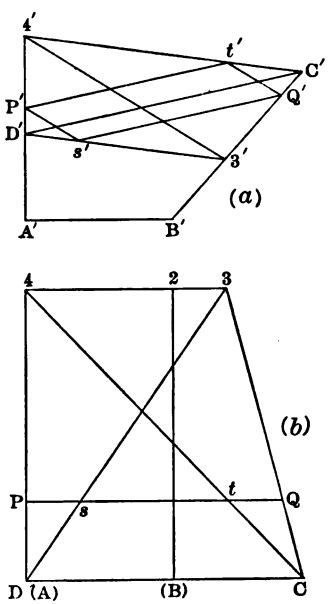

FIG. 27.

diagonal in some point t (Fig. 27). Project the end areas $PQ67$ and 1234 on some vertical plane parallel to them. The point t will be

projected at t', Fig. 27 (a), on vertical projection $4'C'$ of diagonal $4C$. The line tQ is parallel to 34 in Fig. 27 (b), and Pt is parallel to DC. The lines $Q't'$ and $P't'$ will therefore be parallel to $3'4'$ and $D'C'$ respectively.

From similar triangles CtQ and $C34$ we have

$$tQ : 34 : : x : 1,$$

where x = altitude of CtQ = dist. of plane $PQ67$ from $ADBC$.

From similar triangles $DC4$ and $P4t$ we have

$$Pt : DC : : 1 - x : 1.$$

Now if the upper surface is determined by two planes $D3C$ and $D34$, the plane $PQ67$ will cut diagonal $D3$ at same point $s(s')$.

From similar triangles DPS and $D34$, and $3SQ$ and $3DC$ we have

$$PS : 43 : : x : 1;$$

$$SQ : DC : : 1 - x : 1.$$

The lines DC, tQ, tP, sQ, sP, and 34 are equal to their vertical projections $D'C'$, $t'Q'$, etc., as these lines are all parallel to the vertical plane. We can therefore write

$$t'Q' : 3'4' : : x : 1;$$

$$p'S' : 3'4' : : x : 1.$$

Hence $t'Q$ is parallel and equal to $P'S'$. The figure $Q't'P'S'$ is therefore a parallelogram in all positions of PQ. If the surface is generated by a straight line PQ, the volume determined by considering the surface composed of the planes $C43$ and $C4D$ will be in excess by a volume that will be generated by triangle $Qt'P'$. The volume calculated by using diagonal $D3$ will be less than that determined by the ruled surface by a volume that will be generated by triangle $P'S'Q'$. But triangle $t'P'Q'$ = triangle $Q'S'P'$. As these two triangles will be equal in every position of PQ, it is clear that the true volume will be an arithmetical mean between the two volumes found by considering the upper surface determined by the diagonals.

The volume found by using diagonal $C4$ is

$$\tfrac{1}{6}(c_1d_1 + c_2d_2 + c_2d_1 + ah_1 + ah_1 + ah_2).$$

That found by using diagonal $D3$ is

$$\tfrac{1}{6}(c_1d_1 + c_2d_2 + c_1d_2 + ah_2 + ah_2 + ah_1).$$

The true volume is found by taking the mean.

$$\therefore \text{ true volume} = \frac{1}{6}\left[c_1d_1 + c_2d_2 + \frac{c_1d_2}{2} + \frac{c_2d_1}{2} + \frac{3a}{2}(h_1 + h_2) \right] = V.$$

When $PQ67$ is midway between $ABCD$ and 1234 we have

$$c_{\frac{1}{2}} = \tfrac{1}{2}(c_1 + c_2);$$
$$h_{\frac{1}{2}} = \tfrac{1}{2}(h_1 + h_2).$$

$$S_{\frac{1}{2}} = \tfrac{1}{2}(c_{\frac{1}{2}}d_{\frac{1}{2}} + \tfrac{1}{2}ah_{\frac{1}{2}})$$

$$= \tfrac{1}{8}(c_1 + c_2)(d_1 + d_2) + \frac{a}{4}(h_1 + h_2),$$

or $$4S_{\frac{1}{2}} = \tfrac{1}{2}(c_1 + c_2)(d_1 + d_2) + a(h_1 + h_2);$$

$$B_1 = \tfrac{1}{2}c_1d_1 + \tfrac{1}{2}ah_1;$$

$$B_2 = \tfrac{1}{2}c_2d_2 + \tfrac{1}{2}ah_2.$$

$$\therefore V = \tfrac{1}{6}(B_1 + 4S_{\frac{1}{2}} + B_2) = \frac{1}{6}\left[c_1d_1 + c_2d_2 + \frac{c_1d_2}{2} + \frac{c_2d_1}{2} + \frac{3a}{2}(h_1 + h_2)\right].$$

But this is the same result that was found for the true volume as in Art. 35. Thus the prismoidal formula of Newton determines the volume when the upper surface is a warped surface (hyperbolic-paraboloid).

The mean areas and volumes for the example in Art. 35, when the upper surfaces on right and left of centre are warped are given in the following table :

Station.	Areas.	L. C.	C. C.	R. C.	Areas.
1	297	$\frac{18}{36}$	12	$\frac{10}{24}$	189
(+ 50)	(348)	$\left(\frac{19}{37.5}\right)$	(14)	$\left(\frac{12}{27}\right)$	(243)
2	402	$\frac{20}{39}$	16	$\frac{14}{30}$	303

$$M' = \tfrac{1}{8}(297 + 4 \times 348 + 402) = 348.5;$$
$$M = \tfrac{1}{8}(189 + 4 \times 243 + 303) = 244.$$

$\therefore$ total mean area = 592.5;
volume = 2194.44 cu. yds.

37. The whole section $PQ67$ (Fig. 26) may be conceived as moving from $ABCD$ to 1234.

Let x = dist. from $ABCD$ to moving section $PQ76$.

In Fig. 28 let $DA = c_1$, $14 = c_2$, $48 = c_2 - c_1$, where $D8$ is parallel to $1A$.

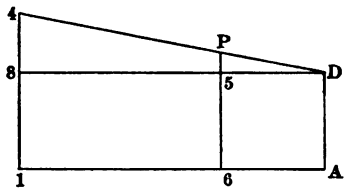

FIG. 28.

From similar triangles $D5P$ and $D48$ we get

$$P5 = \frac{48 \times D5}{D8} = (c_2 - c_1)\frac{x}{l}.$$

$$\therefore P6 = c_1 + \frac{x}{l}(c_2 - c_1).$$

In the same way we can find side height (h_x) at Q:

$$h_x = h_1 - \frac{x}{l}(h_1 - h_2).$$

Now $B_x = \frac{1}{2}c_x d_x + \frac{1}{2}ah_x$ = area $PQ76$.

But $d_x = a + sh_x = a + sh_1 - \frac{sx}{l}(h_1 - h_2) = d_1 + \frac{sx}{l}(h_2 - h_1)$.

$$\therefore B_x = \frac{1}{2}\left(c_1 + \frac{x}{l}(c_2 - c_1)\right)\left(d_1 - \frac{sx}{l}(h_2 - h_1)\right) + \frac{1}{2}a\left(h_1 + \frac{x}{l}(h_2 - h_1)\right)$$

$$= \tfrac{1}{2}c_1 d_1 + \tfrac{1}{2}ah_1 + \frac{x}{2l}[d_1 x(c_2 - c_1) + (h_2 - h_1)(c_1 + a)] + \frac{sx^2}{l}(h_1 - h_2)(c_1 - c_2).$$

38. If in Fig. 14 the side slopes CO and FO make an angle 2θ, that is, if angle FOC of the grade triangle equals 2θ, we have

$$\tan\theta = s;$$

$$\sec\theta = (1 + s^2)^{\frac{1}{2}};$$

$$\cos\theta = \frac{1}{(1 + s^2)^{\frac{1}{2}}};$$

$$\sin\theta = \frac{s}{(1 + s^2)^{\frac{1}{2}}};$$

$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{2s}{1 + s^2}.$$

39. Let Fig. 29 be a two-level railroad cut whose upper surface is a hyperbolic-paraboloid. Project the upper base $ABCD$ orthogonally

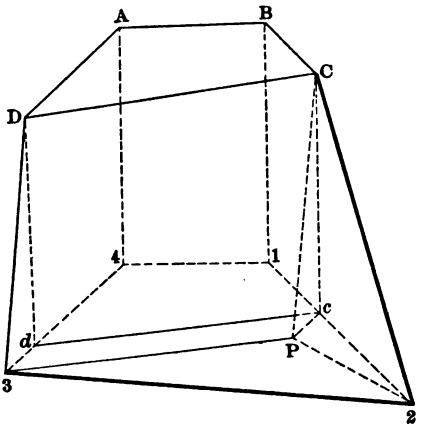

FIG. 29.

on the lower in $14cd$. Draw $3P$ parallel and equal to dc. Join $P2$ and PC. The triangle $cP2$ is the base of the associate pyramid. Let the side heights at 2, 3, C, and D be h_1, h_1', h_2, and h_2' respectively.

Then
$$12 = h \sec\theta = h:(1 + s^2)^{\frac{1}{2}};$$
$$34 = h_1'(1 + s^2)^{\frac{1}{2}};$$
$$BC = 1c = h_2(1 + s^2)^{\frac{1}{2}};$$
$$DA = d4 = h_2'(1 + s^2)^{\frac{1}{2}}.$$

Hence
$$c_2 = 21 - 1c = (h_1 - h_2)(1 + s^2)^{\frac{1}{2}};$$
$$3d = 34 - 4d = (h_1' - h_2')(1 + s^2)^{\frac{1}{2}};$$
$$A = \text{area } cP2 = \tfrac{1}{2}cP \times cs \sin Pc2.$$

Angle $cP2$ is equal to the vertical angle of the grade triangle $= 2\theta$.

$$A = \tfrac{1}{2}(h_1 - h_2)(h_1' - h_2')(1 + s^2) \cdot \frac{2s}{1 + s^2} = s(h_1 - h_2)(h_1' - h_2').$$

But from Art. 25 we have

$$B_1 = sh_1h_1' + a(h_1 + h_1');$$
$$B_2 = sh_2h_2' + a(h_2 + h_2');$$
$$4S_{\frac{1}{2}} = s(h_1 + h_2)(h_1' + h_2') + 2a(h_1 + h_2 + h_1' + h_2').$$

$$\therefore M = \tfrac{1}{6}(B_1 + 4S_{\frac{1}{2}} + B_2)$$
$$= \tfrac{1}{6}[2sh_1h_1' + 2sh_2h_2' + sh_2h_1 + sh_1h_2' + 3a(h_1 + h_2 + h_1' + h_2')].$$

Let $M_e = \frac{1}{2}(B_1 + B_2) = \frac{1}{2}[sh_1h_1' + sh_2h_2' + a(h_1 + h_1' + h_2 + h_2')]$.

$$\therefore M_e - M = \frac{a}{6}(h_1 - h_2)(h_1' - h_2') = \tfrac{1}{6}A. \quad . \quad . \quad . \quad (14)$$

40. Five-level or Irregular Ground.—When the nature of the ground demands it levels must be taken at intermediate points on

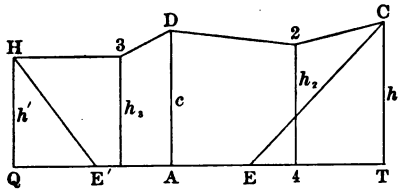

FIG. 30.

the cross-section. If the intermediate levels are only two, the ground is usually classed as five-level; if more, as irregular. The co-ordinates of every point on the surface where it changes slope are given with reference to A as an origin in the "heights" and "distance-cuts" of the note-book. The coordinates of any point are, as explained in Art. 23, written in the form of a fraction, where the numerator represents the ordinate (height) and the denominator represents the distance out from the centre line AD. The area can easily be found from the figure by summing up the areas of the trapezoid and subtracting from the gross sum the areas of the two triangles $HE'Q$ and CET. The usual data given are arranged as follows:

Station.	L. C.	C. C.	R. C.
1	$\frac{h'}{d'}$ $\frac{h_2}{d_2}$	c	$\frac{h_2}{d_2}$ $\frac{h}{d}$

Now the double area of $E'EC_2D_3H = 2(HQTC_2D_3) - 2(HE'Q) - 2(CET) = (h' + h_3)(d' - d_3) + (h_3 + c)(d_3 - o) + (c + h_2)(d_2 - o) + (h_2 + h)(d - d_2) - sh'^2 - sh^2$.

The analogy of this method of finding the area to the usual rule in surveying by the use of the double meridian distances is apparent at a glance. If we take QT as the north and south line, the expressions $(h' + h_3)$, $(h_3 + c)$, etc., are the D.M.D.'s, while $(d' - d_3)$ $(d - d_2)$

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are the latitude difference. Again, if we refer all points on the surface to A as origin, we could find the double area by multiplying each height (ordinate) by the difference of the alternate abscissas taken in order and summing the products.

PROBLEM 6. Find the volume of the following cut when $1 = 100$, $2a = 20$, $s = 1:1$ when the surface lines are those indicated by the lines joining the fractions:

Station.	L. C.	C. C.	R. C.
1	$\frac{10}{20}$ $\frac{16}{10}$	14	$\frac{18}{10}$ $\frac{24}{20}$ $\frac{20}{30}$
2	$\frac{20}{30}$ $\frac{24}{20}$ $\frac{20}{10}$	22	$\frac{28}{10}$ $\frac{30}{20}$ $\frac{36}{30}$ $\frac{40}{50}$

PROBLEM 7. The following notes were taken to find the amount of filling necessary to raise an old road-bed to a higher level. Find volume in cubic yard by average end areas.

Station.	L. C.	C. C.	R. C.
1	$\frac{15}{24}$ $\frac{5}{9}$	4.5	$\frac{4}{9}$ $\frac{12}{21}$
2	$\frac{18}{27}$ $\frac{7}{9}$	6	$\frac{5}{9}$ $\frac{15}{24}$

41. In passing from excavation to embankment the point of intersection of centre line of road-bed with the ground surface is called the grade point. It is necessary to establish this.

Let x = distance from last centre cut to grade point;

c_1 = centre cut at last section in excavation;

c_2 = centre fill at first section in embankment;

l = distance between sections.

Then $$x : l - 4 :: c_1 : c_2,$$

or $$x : l :: c_1 : c_1 + c_2.$$

$$\therefore x = \frac{c_1 l}{c_1 + c_2}.$$

42. The width of road-bed in fills is often made less than the width in the adjacent cuts, as no ditches for drainage have to be provided for

in fills. If the width of road-bed in cuts is 18 feet, that in fills under three feet in depth is often made 12 feet; and where depth of fill is over three feet, the width is made 14 feet. Not only the grade point should be located, but the points where the edges of road-bed cut the

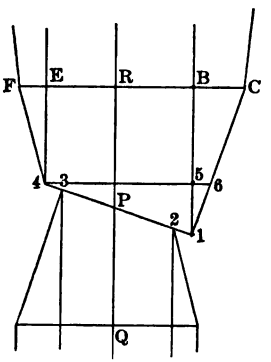

FIG. 31.

surface should be located. If the ground is sloping uniformly, the line of intersection of the road-bed plane with the terrain will be a straight line. If levels are taken at B and E and at other points between cross-section FC and grade line 14, we can find the distances $B1$ and $E4$ by calculation. For ordinary three-level ground $B1$ and $E4$ will have approximately the same slope that RP has. In irregular ground the points on the grade line would have to be located by trial.

The volume of the excavation between FC and 14 will have to be determined by locating a section 456 through 4. The sum of the volumes of the prismoid 456-FC and of the prism 1-456 will give the true volume of excavation.

Example. — Given $2a = 20$, and at station R the notes are $\left(\frac{3}{13}, 6, \frac{11.2}{21.2}\right)$, and RP, $B2$, and $E4$ were found to be respectively 45, 48, and 42 feet.

Area of cross-section at $FC = 173.6$ square feet. The coordinates of cross-section through 4 are found by interpolation to be $\left(\frac{0}{10}, .4, \frac{1.4}{11.4}\right)$, and the area of the cross-section is 11.28 square feet. The coordinates of the mid-section are $\left(\frac{1.5}{11.5}, 3.2, \frac{6.3}{16.3}\right)$, and its area is 83.48 square feet.

$$\text{Volume} = \frac{42}{6 \times 27}(173.6 + 4 \times 83.48 + 11.28) = 134.5 \text{ cubic yards.}$$

Volume of prism $= \frac{6}{3 \times 27}(11.28) = .84$ cubic yards.

Total volume = 135.34 cubic yards.

PROBLEM 8. Find volume in cubic yards by average end areas by calculation, both in excavation and embankment, from following notes:

Station.	L. C.	C. C.	R. C.
0	$\frac{20}{39}$	18	$\frac{14}{30}$
1	$\frac{16}{33}$	14	$\frac{10}{24}$
2	$\frac{12}{27}$	10	$\frac{7}{19.5}$
3	$\frac{8}{21}$	6	$\frac{4}{15}$
4	$-\frac{2}{10}$	−4	$\frac{6}{16}$

PROBLEM 9. Find volume in Problem 8 by prismoidal formula.

PROBLEM 10. Find volume in cubic yards by use of earthwork diagram, in Problems 8 and 9.

43. Borrow-pits.—It is occasionally necessary to borrow earth for fills when it would cost less than hauling it from a distant cut. These borrow-pits are localities or areas not in the line of the graded road or areas near the proposed fill from which earth can be obtained more economically than from some cut in the line of road. It occasionally happens that the excavations do not contain enough earth to construct the embankment, and of course earth has to be borrowed to complete the fill. If it is cheaper to borrow earth than to haul it from a distant cut, the surplus earth taken out of the cut is "wasted."

The earth is measured in place, and for this purpose the ground must be divided into polygonal areas. Levels are then taken at the vertices of these polygons both before and after excavating. Some method of re-establishing these points for the purpose of taking the second series of levels must be adopted. This can be most easily done by dividing the pit into rectangular areas as in Fig. 32, and by placing guide-stakes in the prolongation of the dividing lines at a certain recorded distance from the edge of the pit on all four sides. It is really only necessary to re-establish one known line as AB and mark on it the points 1, 2, 3, 4, etc. Then the lines at right angles to AB can be located, and the other points as a, b, c, and d

can be re-established. After all division points in the pit have been relocated, the second series of levels are taken.

Various methods of keeping the notes for borrow-pits are used. A very convenient method is shown in the following table. A page of

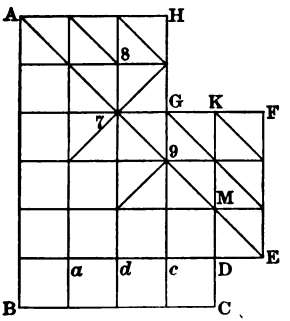

FIG. 32.

the ordinary field-book is ruled into as many vertical columns as there are ranges. Thus if *ABCDEFGH* is a borrow-pit we should adopt *AB* as our reference line, and in the station column we record the number (1, 2, 3, etc.), and to the right of these stations we record the level-readings on the different points in the column headed by its distance from *AB*. In the table the squares are supposed to be 10 × 10 ft.

Station.	0	10	20	30	40	50
0	23 / 9	25 / 9	27 / 9	29 / 9		
1	21 / 10	23 / 10	25 / 10	27 / 10		
2	19 / 10	21 / 11	23 / 11	25 / 11	27 / 10	29 / 9
3	17 / 10	19 / 11	21 / 11	23 / 11	25 / 10	27 / 9
4	15 / 10	17 / 10	19 / 10	21 / 10	23 / 10	25 / 10
5	13 / 10	15 / 10	17 / 10	19 / 10	21 / 10	23 / 10
6	11 / 10	13 / 9	15 / 9	17 / 9	19 / 9	

The first reading on the points are given by the numerators and the second readings are the denominators. The readings are the "elevations" as referred to the same datum-plane. The heights or depths of cuts are given by their difference.

The volume, if the upper surface over the small rectangular is a plane, can be accurately found by

$$V = \frac{a^2}{6}(2h_1 + h_2 + 2h_3 + h_4),$$

where h_1 and h_3 are the heights at tne ends of the dividing diagonal EM. Again, we have

$$V = \frac{a^2}{6}(2h_2 + 2h_4 + h_1 + h_3).$$

Adding, we get

$$2V = \frac{a^2}{6}(3h_1 + 3h_2 + 3h_3 + 3h_4).$$

$$\therefore V = \frac{a^2}{4}(h_1 + h_2 + h_3 + h_4). \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Thus the solid can be treated as a truncated prism, and its volume is equal to a prism of the same base and whose altitude is one-fourth of the four corner heights. Now some of the corner heights (as at $AHFEc$ and B) will enter as factors only once, some twice (as those at intermediate points on sides as 1234, etc.), some three times (as at D and G), and some four times (as at M and all other points not on the boundary). By comparing Fig. 32 and the table we see that the heights that enter only once as factors are those in the table that have a vacant square both in column and row adjacent; those that enter twice have a vacant square either in column or row; those that enter three times as factors have all spaces full except one on a diagonal; and those that enter as factors four times are completely surrounded. For the special example in the table, it is supposed that the pit is graded down to a level plane whose height is 10 feet above datum.

$$\Sigma h_1 = 13 + 1 + 9 + 13 + 19 + 19 = 74;$$

$$2\Sigma h_2 = 2(15 + 17 + 17 + 17 + 15 + 7 + 5 + 3 + 3 + 5 + 7 + 9 + 11) = 262;$$

$$3\Sigma h_3 = 3(15 + 11) = 78;$$

$$4\Sigma h_4 = 4(13 + 15 + 11 + 13 + 9 + 11 + 13 + 15 + 7 + 9 + 11 + 13 + 5 + 7 + 9) = 644.$$

$$V = \frac{a^2}{4 \times 27}(74 + 262 + 78 + 644) = 979.63 \text{ cu. yds.}$$

If the ground is rolling and not very uniform in fall, the sides of the square or rectangles must be taken small. It will greatly simplify the calculation if the dimensions of these rectangles are multiples of 9. Thus if $a = 18$ feet in the example given, we have

$$V = \frac{18 \times 18}{4 \times 27}(1058) = 3174 \text{ cu. yds.}$$

If * the ground is of such a character that we can divide the surface into rectangles $30' \times 36'$, we have

$$V = \frac{30 \times 36}{4 \times 27}(\Sigma h_1 + \Sigma h_2 + \Sigma h_3 + \Sigma h_4)$$

$$= 10(\Sigma h_1 + \Sigma h_2 + \Sigma h_3 + \Sigma h_4). \quad . \quad . \quad . \quad . \quad . \quad (15a)$$

The formulæ for borrow-pits apply with equal facility to the problem of finding the volume of earth removed in grading down elevations to a common level or to any desired surface, and to finding the volume of earth removed in excavating for reservoirs, for the foundations for buildings, etc.

Where it is apparent that the upper surface of any of the truncated prisms is not a plane, a closer approximation to the true volume can be found by dividing the rectangles or squares in the plan of the network of the borrow-pit by diagonals, thus using triangles as the bases of the prisms. The engineer will, when laying out the first network of rectangles, indicate in the table or on the drawing of the plan the diagonal that will most nearly lie on the surface of the earth, and divide the given upper surface into two planes whose intersection is this diagonal. Usually these diagonals will have one direction, but in very rough ground they may follow no law, and each diagonal will have to be drawn independent of the others. Thus it may happen that the height at a corner will be in only one triangle as at F, in two as at E, in three as at K, in four as at G, in five as at 8, in six as at M, in seven as at 9, and in eight as at 7. Now the volume of any triangular prism whose base is A and whose heights are h, h', and h'' is given by

$$V = \frac{A}{3}(h + h' + h'').$$

If the sides of the rectangle are a and b, the complete volume in cubic yards will be given by

$$V = \frac{ab}{6 \times 27}(h_1 + 2\Sigma h_2 + 3\Sigma h_3 + \ldots + 7\Sigma h_7 + 8\Sigma h_8),$$

where $h_1, h_2, \ldots h_8$ have the same significance as in formula (15a).

* See J. B. Johnson's Surveying.

The direction of the diagonals can be indicated in the table by joining the proper fractions as shown in the table by the lines joining 17 and 21 in the first and second columns.

44. Cost of Earthwork.*—It is usual to pay for earthwork by the cubic yard in excavation instead of allowing separate prices for excavation and embankment. Of course, when borrow-pits are used to construct an embankment it is allowed for as usual, but is estimated in cut at the pit. The method of measuring and estimating at cut is more expeditious and is liable to prevent a misunderstanding between engineer and contractor, as it is difficult to agree upon the amount of reduction that should be made in embankment for shrinkage.

The cost of excavating and removing earth is made up of the following items:

a. Loosening.
b. Loading into carts and barrows.
c. Hauling.
d. Spreading.
e. Keeping road in repair.
f. Wear of tools, etc.
g. Superintendence and water-carriers.
h. Profit of contractor.

45. Loosening the earthwork for the shovellers can be done by the plow or pick. Two good men with two good horses and a plow can loosen in heavy soils from 200 to 300 cubic yards per day, and in loam they can loosen from 400 to 600 cubic yards per day. The cost of such work will vary with locality and with the current prices of labor. Assuming $1 per day for men, 75 cents per horse, and 37 cents for the plow (which is excessive), we have $3.87 as the daily cost of the outfit. The cost per yard for loosening will therefore be 1.93 to 1.29 cents in heavy soils and .98 to .64 cents in common loam per cubic yard. We can economize by using four-horse plows instead of two-horse. A four-horse team can be managed by one driver and, allowing one man to the plow, the total cost per day will be $5.37. Such a team can loosen 400 to 600 cubic yards of heavy soil and 800 to 1200 yards of common loam. Thus the cost per yard for loosening can be reduced to 1.34 to .90 cents for heavy soils and to .67 to .45 cents per yard in loam.

A fairly good hand can, with a pick, loosen 14 yards of pure stiff clay, 25 yards of strong heavy soils, and 40 yards of common loam, and 60 yards of light sandy soils. At $1 per day the prices per yard will be $7\frac{1}{7}$ cents for stiff clay, 4 cents in strong heavy soils, $2\frac{1}{2}$ cents in common loam, and $1\frac{2}{3}$ cents in light sandy soils.

* The data for the articles on cost of earthwork have been taken with few changes from Trautwine.

46. If the earth is removed by carts, shovellers have to be provided. The amount a man can shovel a day into carts depends on many circumstances, weight of material, the time lost in waiting for carts, etc. A good manager can adjust matters so that the shovellers need lose no time, for, should there be no cart ready to be filled, the shovellers can help the pickers. In fact, it would save time and would conduce to economy to have pickers and shovellers to work together as far as possible. When there is no cart to fill, all can use the picks, and as soon as the cart backs in the shovellers should be assisted by at least two pickers, thus reducing the idling time of the horse to a minimum. A shoveller can load $\frac{1}{15}$ of a cubic yard of sandy soil (measured in place) per minute, $\frac{1}{18}$ of a cubic yard of loam, and $\frac{1}{21}$ of a yard of heavy soil. Thus in a day a shoveller can handle 40 yards (120 loads) of light sandy soil, $33\frac{1}{3}$ yards (100 loads) of loam, and 28.7 yards (86 loads) of heavy soil. On account of the loss of time Trautwine estimates that $\frac{4}{10}$ of the time of the shovellers will be lost. This reduces the capacity of the shovellers to 24 yards in light sandy soil, 20 yards in loamy soils, and 17.2 yards in heavy soils. Then if the shovellers lose no time, the cost per yard for shovelling will be, with labor at $1 per day, $2\frac{1}{2}$ cents per yard in light sandy soils, 3 cents in loam, and $3\frac{1}{2}$ cents in heavy soils. If $\frac{4}{10}$ of the time is lost, the cost of shovelling will be $4\frac{1}{2}$ cents per yard for sandy soil, 5 cents for loams, and 5.81 cents for heavy soils and clay.

47. Trautwine assumes that a horse travels, while working to a dump-cart, at the rate of $2\frac{1}{2}$ miles per hour or at the rate of 200 feet per minute. A rate of $2\frac{1}{2}$ miles per hour is equivalent to $205\frac{1}{2}$ feet per minute, and a rate of $2\frac{1}{2}$ miles per hour will be equivalent to 198 feet per minute. So that in reality if we assume a rate of 200 feet per minute it is more nearly equivalent to a rate of $2\frac{1}{2}$ miles per hour than $2\frac{1}{2}$.

Let l = lead = distance from pit to dump;

$2l$ = total distance travelled on one trip;

$\frac{2l}{200}$ = time in minutes required to travel $2l$.

About 4 minutes is lost on each trip in dumping and turning.

Total time $t = 4 + \frac{\text{lead}}{100} = 4 + \frac{1}{100}$.

If L = number of loads hauled, we have

$$L = \frac{600}{4 + \frac{l}{100}};$$

and as the capacity of the cart is about ¼ of 1 cu. yd., we have, where N = number of cubic yards,

$$N = \frac{200}{4 + \frac{l}{100}}.$$

For loading loose rock an allowance of six minutes is made (instead of four) for lost time.

$$\therefore \quad n = \frac{200}{6 + \frac{1}{100}}.$$

If we lay off the leads to scale on a horizontal line ox, Fig. 33, and at the different points erect lines perpendicular to ox, and on these lay off to scale the values of n from K, we obtain the curve marked "carts" in Fig. 33. Now for any lead whatever (within the limits of the figure) we follow the vertical line through the lead (on ox) to the curve, and from this point of intersection we follow the horizontal line to line oy and read off the number of yards. Thus for a lead of 450 we follow the vertical through 450 to intersection point and then follow the horizontal line through this point to line oy, and read off the number of yards 23.5. While the curve in the diagram is constructed for leads from 0 up, it is not supposed that leads of very small amounts will ever be demanded. Other curves are laid off in Fig. 33, to which reference will be made later.

48. Spreading or levelling the embankment is always necessary where the embankment is intended for construction purposes. A laborer can spread 50 to 100 yards per day of 10 hours, and this will cost on an average, with labor at $1 00 per day, 1.5 cents per yard.

49. For the repair of roads an allowance of $\frac{1}{1000}$ cent per yard is made.

50. The allowance for wear, sharpening tools, superintendence, and water-carriers, for the extra trouble of levelling off cut, digging side ditches, and trimming up, is about 2 cents per yard.

51. By using the foregoing data the following general expression (p. 62) for the cost per cubic yard for various leads can be found.

If the earth is not spread into layers, but is "wasted," only ¼ of a cent per yard is allowed for "spreading" instead of 1.50. This ¼ of a cent is allowed for keeping the dump in order. Then, if the earth is wasted, we have, for the four classes of soils:

$$\text{For class 1 the cost per yard} = 12.25 + \frac{116l}{16000};$$

For class 2 the cost per yard $= 14.56 + \frac{116l}{16000}$;

" " 3 " " " " $= 16.56 + \frac{116l}{16000}$;

" " 4 " " " " $= 10.08 + \frac{116l}{16000}$.

SUMMARY BY CARTS.

Labor $1 per Day of 10 Hours.

Items.	Class I. Common Loam.	Class II. Strong, Heavy Soils.	Class III. Pure Stiff Clay or Cemented Gravel.	Class IV. Light Sandy Soil.
Loosening (pick)	2.50	4.00	7.00	1.67
Loading	5.00	5.81	5.81	4.16
Spreading	1.50	1.50	1.50	1.00
Wear	2.00	2.00	2.00	2.00
Hauling	$2.50 + \frac{l}{160}$	$2.50 + \frac{l}{160}$	$2.50 + \frac{l}{160}$	$2.50 + \frac{l}{160}$
Repair of road	$\frac{l}{1000}$	$\frac{l}{1000}$	$\frac{l}{1000}$	$\frac{l}{1000}$
Total cost per yard	$13.50 + \frac{116l}{16000}$	$15.81 + \frac{116l}{16000}$	$18.81 + \frac{116l}{16000}$	$11.33 + \frac{116l}{16000}$

If the earth is loosened by plows instead of picks, the cost of loosening per yard will be only $\frac{1}{8}$ of a cent instead of $2\frac{1}{2}$, and the cost per yard will be as follows:

For loam (plowed), cost per yard $= 11.80 + \frac{116l}{16000}$;

" strong heavy earth, " " " $= 14.11 + \frac{116l}{16000}$.

Now, if the earth is plowed and wasted a further reduction of 1.25 cents per yard will occur, and we shall have,

For loam (plowed and wasted), cost per yard $= 10.50 + \frac{116l}{16000}$;

" heavy soil (plowed and wasted), " " " $= 12.86 + \frac{116l}{16000}$.

Trautwine assumes that a man with a wheelbarrow will move at the same rate that horse to cart does, namely, 200 feet per minute. If $1\frac{1}{2}$ minutes be allowed for lost time in loading, etc., we have

$$t = 1.25 + \frac{l}{100}.$$

If the man is continuously at work for $\frac{9}{10}$ of his time, *i.e.*, $\frac{9}{10} \times 600 = 540$ minutes, we have

$$L = \frac{540}{1.25 + \frac{l}{100}} = \text{number loads per day.}$$

Each barrow holds about $\frac{1}{14}$ of a yard.

$$n = \frac{540}{14\left(1.25 + \frac{l}{100}\right)}.$$

In loose rock an allowance of 1.6 minutes is made for lost time instead of 1½. If labor is worth $1 per day and an allowance of 5 cents is made for the wear of wheelbarrow, we have

$$\text{Cost of hauling per yard} = \frac{105}{540} \times 14\left(1.25 + \frac{l}{100}\right)$$

$$= 3.40 + \frac{49l}{1800}.$$

52. The following summary can be verified by reference to the appropriate sections and "items":

SUMMARY BY WHEELBARROW.

Labor at $1 per day of 10 hours.

$$y = \frac{508l}{18000}.$$

Items.	Common Loam.	Straight Heavy Soil.	Pure Stiff Clay.	Light Sandy Soil.
Loosening (pick).....	2.50	4.00	7.00	1.67
" (plow).....	1.80	1.60	2.50	.40
Spreading	1.50	1.50	1.50	1.00
Wasted.....	.25	.25	.25	.25
Wear, superintendence, etc.	2.00	2.00	2.00	2.00
Hauling	$3.40 + \frac{49l}{1800}$	$3.40 + \frac{49l}{1800}$	$3.40 + \frac{49l}{1800}$	$3.40 + \frac{49l}{1800}$
Repair of road.....	$.001l$	$.001l$	$.001l$	$.001l$
Total cost per yard picked and plowed.....	$9.40 + y$	$10.90 + y$	$13.90 + y$	$8.07 + y$
Total cost per yard picked and wasted	$8.15 + y$	$9.65 + y$	$12.65 + y$	$7.32 + y$
Total cost per yard plowed and spread	$7.70 + y$	$8.50 + y$	$9.40 + y$	$6.80 + y$
Total cost per yard plowed and wasted	$6.45 + y$	$7.25 + y$	$8.15 + y$	$6.05 + y$

53. Drag Scrapers.—A drag scraper holds about $\frac{1}{2}$ of a cubic yard and the team moves at the rate of 150 feet per minute. The team

cannot travel on a direct line between pit and dump, and the lead is increased by 15 feet to allow for this. Then we have

$$t = \frac{2(l+15)}{150} = \text{time in minutes for one trip;}$$

$$L = \frac{600}{t} = \frac{600 \times 150}{2(l+15)} = \frac{300 \times 150}{l+15} = \frac{45000}{l+15};$$

$$n = \frac{9000}{l+15}. \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (15c)$$

54. A curve is constructed for equation (15*c*) as the "carts" curve was constructed in Fig. 33. This curve for (15*c*) is laid on some diagram and is marked "drag." But in using Fig. 33 for drag or wheeled scrapers we must be careful to read the number of yards from the proper side. For drag and wheeled scrapers follow the vertical line to the proper curve and then follow the horizontal line through this point to the volume line on the *right*.

Wheeled scrapers have the advantage of offering less resistance to traction and they can be safely utilized on longer hauls. The box is made of sheet steel and is about $3\frac{1}{2} \times 3\frac{1}{2} \times 1\frac{1}{2}$ feet and contains, when full, about .56 of a cubic yard. The scrapers are generally filled and emptied by special laborers at the pit and dump. The team is supposed to be kept going all the time and, as in case of drag scrapers, is supposed to travel at the rate of 150 feet per minute. Trautwine limits the amount hauled per trip as follows:

For leads less than 100 ft.	.33	cubic yard
For leads between 100 and 300 ft.	.40	" "
For leads between 400 and 500 ft.	.50	" "
For leads over 500 ft.	.60	" "

It is supposed that the team travels 25 feet more than the lead on account of the necessity of keeping out of the way of the other teams and of making a partial circuit both at the dump and pit. Then using the same notation as before we have

For leads less than 100 ft. $\ldots\ldots\ldots

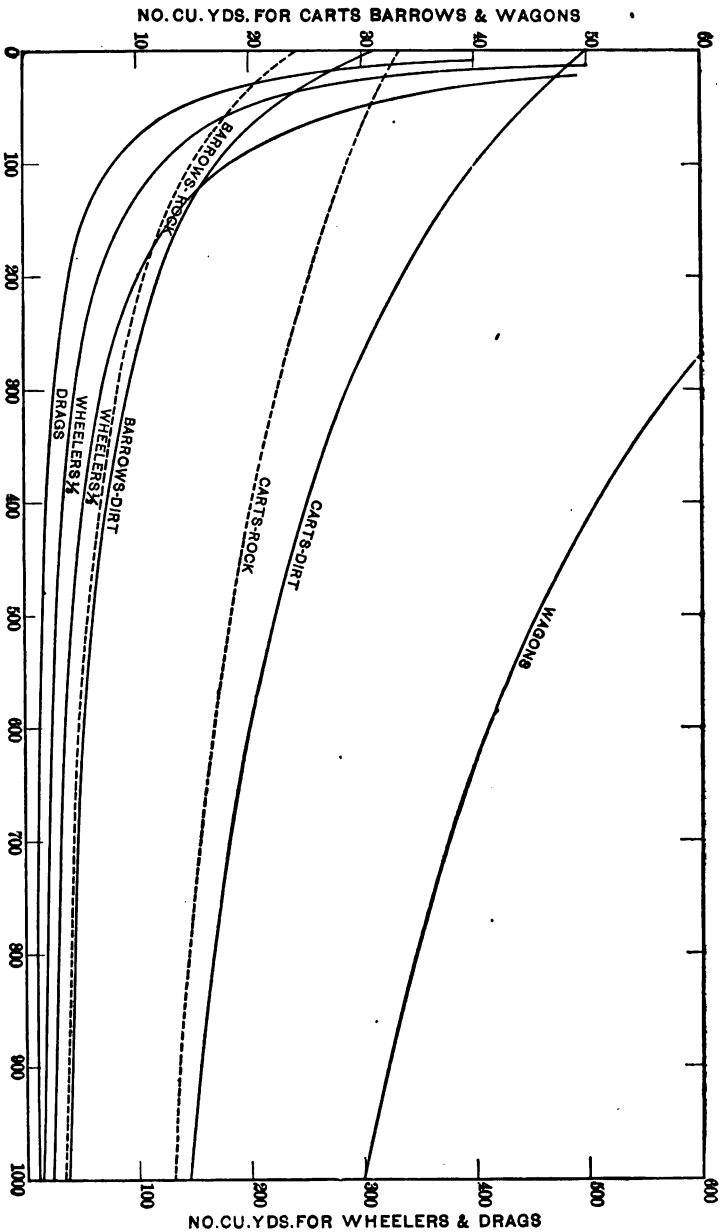
NO. CU. YDS. FOR CARTS BARROWS & WAGONS
10
20
30
40
50
60
0
100
200
300
400
500
600
700
800
900
1000
LEADS
BARROWS-ROCK
BARROWS-DIRT
WHEELERS ½
WHEELERS ¼
DRAGS
CARTS-ROCK
CARTS-DIRT
WAGONS
100
200
300
400
500
600
NO. CU. YDS. FOR WHEELERS & DRAGS

FIG. 33.

horses usually hauls about ⅞ to 1⅛ of a cubic yard at a load, and with such a load can travel about 180 feet per minute. The wagons are loaded at a "trap," which consists of a floored bridge with an aperture across a cut 8 to 10 feet wide, and deep enough for wagons to pass under, or a platform, constructed of logs or framing high enough for a wagon to pass under, which is reached and left by inclined roadways. The wagon is driven under the bridge and stopped, and the scrapers are driven across the trap and dumped as they pass the opening, delivering their load in the wagon below. It requires good management to prevent great loss of time in having one wagon standing idle while the one ahead is loading.

About five drag-scrapers are used to a trap. The earth is scraped up if of sand or soft loam, otherwise a two-horse, three-horse, or four-horse plow is used according to the toughness of the soil. One plow can readily keep five or six scrapers supplied in light soils. Contractors estimate that one scraper for ordinary leads can deliver 40 cubic yards or 200 loads per day of 10 hours. The time required to load one wagon will be 2½ (say 2¾) minutes. If l = length of lead in feet, the total path described cannot be less than $(2l + 90)$ feet. While one wagon is loading, the one in front travels a distance of $180 \times 2\frac{1}{2} = 450$ feet. The wagons in motion should be this far apart in time. The time of unloading and rearranging the wagon-body will be about two minutes.

If t = time in minutes of making complete round trip, we have

$$t = 2\tfrac{1}{2} + 2 + \frac{2l + 90}{180} = 5 + \frac{l}{90}.$$

If no time is lost, no wagon should be within 450 feet of another. Counting the filling as 450 feet and the dumping 360 feet, the number of wagons necessary will be $\frac{2t}{5} = 2 + \frac{l}{225}$. Then for leads of 225, 450, 675, 900, etc., feet three, four, five, six, etc., wagons are needed.

If N = number of cubic yards that one wagon can haul per day of 10 hours, we have

$$N = \frac{8}{10} \times \frac{600}{5 + \frac{l}{90}} = \frac{480}{5 + \frac{l}{90}}.$$

From the "wagon" curve, Fig. 33, we can read off the number of cubic yards a wagon can haul per day for various leads.

56. Rock.—Hard rock weighs from 150 to 168 pounds per cubic foot. A cubic yard will therefore weigh 150×27 (4050 pounds) or 27×168 (4536 pounds), that is, about 2 to 2.26 short tons, or about 1 8 to 2 long tons. A solid cubic yard of hard rock when broken up

by blasting so that it can readily be handled by wheelbarrows occupies a space of $1\frac{1}{8}$ cubic yards. A cubic foot of the loosened earth will weigh about 90 pounds. Now a wheelbarrow's capacity by volume is about $\frac{1}{14}$ of a cubic yard; the weight of full load will be $90 \times 27 \div 14 = 174$ pounds. But $\frac{1}{2\frac{1}{4}}$ of the cubic yard of stone in place is a little less than $\frac{1}{14}$ of the broken-stone yard. Thus it will require 24 loads of about 177 pounds each to haul one cubic yard in place. On account of the *growth* the one yard of solid rock makes 1.8 yards, and as the barrow can haul only $\frac{1}{14}$ of one yard, the capacity of the barrow when measured in terms of the original cubic yard will be $\frac{1}{25}$ (Trautwine gives $\frac{1}{24}$).

The actual cost of loosening in place, including drilling, powder, tools, etc., is about 45 cents per cubic yard. In practice it will range for rock usually classed as "hard" between 30 and 60 cents, depending on superintendence, strike, toughness, hardness, and thickness of stone.

Then the number (n) of cubic yards that can be transmitted by carts or scrapers is as follows:

For barrows ..... $n = \dfrac{540}{24\left(1.6 + \frac{l}{100}\right)}$.

For carts..... $n = \dfrac{1000}{9\left(6 + \frac{l}{100}\right)}$.

57. Excavation by Steam-shovels.—Where the excavation is on a large scale and where temporary tracks are of easy construction, the cost of excavating can be greatly reduced by the use of steam-shovels. Actual experience has shown that the cost is only about $\frac{1}{3}$ to $\frac{1}{4}$ of what it would be by hand labor. There are several types of machines in use, and many of these types are constructed in different sizes, depending on the contents of the dipper, which varies from $1\frac{1}{2}$ to $2\frac{1}{2}$ cubic yards. The weights of machines corresponding to these different sizes vary from 20 to 40 tons. Their capacity of course varies with the kind of material and the time lost in laying new track and moving forward. A good machine without interruptions can deposit into cars four dippers of gravel or sand per minute, or one in every 15 seconds. This would give a daily capacity for the larger machines of 2400 dippers. It is needless to say that this is not realized in practice. Allowing one hour per day for lost time in shifting, waiting for cars, etc., the amount of earth of the different grades the best shovels can handle is given in the following table:*

* Taken from paper by E. A. Hermann, *Engineering News*, June 29, 1893.

Size of Dipper.	Delav in Hours.	Daily Output, in Cubic Yards.					Remarks.
		Sand.	Dry Loam.	Dry Clay.	Damp Clay.	Stiff Clay.	
2½ cu. yds.....	1	2400	2000	1800	1200	800	Good
2½ cu. yds.....	2½	1800	1500	1350	900	600	Average
1¾ cu. yds.	1	1600	1200	1000	800	600	Good
1¾ cu. yds..... ..	2½	1200	900	750	600	450	Average
1 cu. yd.....	1	1000	800	700	500	400	Good
1 cu. yd.....	2½	750	600	525	375	300	Average

The Otis shovel (built by John Souther & Co., Boston) has a record of 38,168 cu. yds. for the month of October, 1894, and for four other months under the same management it averaged 34,000 cu. yds. per month. It is rare that the monthly output falls as low as 10,000 cu. yds. even in the hardest of material that the excavator can work. About 500 cu. yds. per day is the minimum daily output in very hard material; in medium hard about 1000 cu. yds. is a good average, although the Otis, Victor, and many other good machines have much higher records. It is of great advantage to have cars that will contain as many cubic yards as possible, to prevent delays in waiting. The cars generally hold from 3 to 8 cu. yds.

The machine is operated by an engineer ($4 per day), a cranesman ($4 per day), and a fireman ($2 per day), and four to six pitmen (at $1 to $1.25 per day each), if the cut is not deep enough to need polling, i.e., breaking off the edges of the bank that are too high for the shovel to reach. To the cost of labor must be added that of an overseer; and the cost of oil waste, water, coal, interest, and depreciation will run the daily expenses up to something like $30. The actual cost of excavating, then, will vary from 1 to 6 cents per cubic yard under most favorable circumstances. The Michigan Central has reduced the cost of loading gravel ballast to 1.7 cents per cubic yard.

58. **Overhaul.**—In contracts for earthwork it is usual to let them at so much per cubic yard for all hauls under a specified distance, called the "free haul." All material transported beyond this is allowed for at so much per cubic yard for each 100 feet of extra haul. The extra distance is called the "overhaul." This question of overhaul is often a source of dispute between engineers and contractors, and it is of the utmost importance that all specifications in regard to this matter should be clear and explicit.

In Fig. 34, if G and G_1 be the respective centres of gravity of cut and fill, the distance PM is the average haul. It is found most easily by finding the centre of gravity of excavation and embankment as masses or as geometrical solids. In effect average haul is equal to the quotient obtained by dividing the sum of the products of

each yard of material by the distance it is hauled by the number of cubic yards of earth transported. In Fig. 34, ACE represents the profile of the centre surface line and LK represents the grade line. Now if all of the earth that is taken out of cut is taken into the embankment, and the distance (d) between G and G_1 is less than or

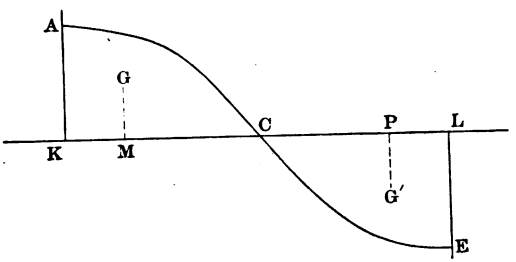

FIG. 34.

equal to the free haul, no extra charge should be allowed for overhaul. The free haul is often as high as 500 feet.

If n = number of cubic yards taken from cut to fill,
d = distance in feet between centres of gravity of cut and fill,
a = free haul in feet,
y = price per cubic yard for each 100 ft. of overhaul,
c = charge for overhaul in cents,

then
$$c = \frac{yn}{100}(d - a).$$

A contract is sometimes written so that the contractor will receive y cents per yard for each additional 100 ft. beyond the free haul, the haul to be measured from centre of gravity of excavation to extreme haul in embankment. With the foregoing notation, the cost per yard at this form of contract would be given by

$$c = \frac{ny}{100}(PL - a),$$

where n = number of cubic yards in CEL.

The additional allowance, then, for this method over the former accurate and proper method is

$$\frac{ny}{100}(PL).$$

If there is more earth in the excavation ACK than is needed for the fill CLE, and if it would not be economical to use it for other

fills, it is wasted. The charges for earthwork are generally arranged under the three items "fills," "wasted," and "overhaul."

59. General Specifications.—A set of specifications should be furnished by the railway company, setting forth the character and quality of the work that will be required before acceptance. These should form the basis for intelligent bidding on the part of the contractors, and should serve as a guide for the engineer in charge, showing him what requirements the work must fulfil. Besides stating that the work must be finished in a "true workmanlike manner, etc.," it should be noted and distinctly understood that the engineer in charge has complete jurisdiction over the work and full authority to decide all questions pertaining to classification, haul, distribution of material, locus of borrow-pits, and on all other questions that may come up for discussion between the contractor and the railway company in regard to the work being done.

(*a*) *Width of Road-bed.*—The road-bed on embankments should be from 12 to 14 feet wide for fills less than 3 feet in height; 14 to 16 feet for fills from 3 to 6 feet in height; and from 16 to 18 feet for fills of greater heights. For cuts up to 6 feet in depth the road-bed should be not less than 18 feet in width; for cuts from 6 to 12 feet in depth, 20 feet in width; for cuts from 12 to 18 feet in depth, 22 feet in width; and above this, 24 feet wide.

(*b*) *Side Slopes.*—The angle of slope of banks depends on the class of material; for fills of ordinary earths, such as sand, loam, and clay, a slope of $1\frac{1}{2}$ horizontal to 1 vertical is usual. For loose rock fills a slope of 1 to 1 is sufficient. For cuts through ordinary earth material a slope of 1 to 1 is usual, except sometimes for sand or very loose earth a slope of $1\frac{1}{2}$ to 1 is necessary. For rock cuts slopes of $\frac{1}{4}$ to 1 and $\frac{1}{8}$ to 1 are sufficient, depending on the class of work.

(*c*) *Classification of Material.*—Material in excavation is classified as "earth," "loose rock," or "solid rock," according to difficulty of removal. Sometimes a classification "gravel" is used as intermediate between earth and loose rock, but this is hardly ever necessary, as the three first mentioned can be made to include all material and justice be done to all parties concerned. "Earth" excavation includes all material that can be readily removed with plow and scraper. "Loose rock" includes all boulders and detached masses of rock from 1 cu. ft. to 1 cu. yd. in dimension, also such material as slate, shale, soft sandstone, "hardpan," and all material that cannot be loosened up with a ten-inch plow behind a good team of six horses or mules well handled. "Hard rock" includes all boulders and detached masses of rock greater than 1 cu. yd. in bulk, all rock in solid beds or masses in its original stratified position, and all material which, in the judgment of the engineer in charge, cannot be removed except by blasting. To avoid controversy on the part of the con-

tractor, it should be distinctly understood and stated that the engineer in charge shall decide on all classification, and he, in his decision, should be governed by the actual difficulty with which the material is removed.

(*d*) *Distribution of Material.*—The material from cuts should be used for making fills up to a point when the amount for overhaul becomes equal to the price per yard; the remainder should be used for widening fills already built, or should be deposited in "spoil-banks" outside of prescribed limits. As far as possible, embankments should be started at the base with the full width indicated by slope-stakes, and be built up to true slope without widening from top.

(*e*) *Overhaul.*—For all material from cuts hauled into embankments beyond a certain distance, termed the "free haul," to the original contract price per yard is added an amount, usually 1 cent per cu. yd. per 100 feet hauled. This excess in distance beyond the "free haul" is termed the "overhaul." Properly defined, the overhaul, as before explained, means the distance from the centre of gravity of the material in the cut to the centre of gravity of the material in embankment, less the "free haul," but its intended meaning should be definitely and clearly set forth in the specifications, as an obscure statement of the terms has been construed to mean the distance from centre of gravity of cut to extreme haul in fill, which almost doubled the customary price for overhaul.

(*f*) *Shrinkage.*—It should be stated that all material will be paid for in "excavation" measurement only; or where measured in embankment, when it is not possible to get a fair estimate from borrow-pits, a certain per cent will be added for shrinkage. This amount, which varies for different material, must be decided by the engineer in charge. For wheelbarrow and cart work from 15 to 20 per cent is added to the original height of bank; for scraper work, from 2 to 5 per cent.

A berm of not less than 6 feet in width should be left undisturbed along all embankments. The borrow-pits from edge of berm should have a slope equal to the slope of the embankment. Borrow-pits should be dug so that they may be easily drained.

60. **Profiles.**—Profiles are plotted on special paper, ruled for the purpose. There are three styles, plates A, B, and C, differing in number of horizontal and vertical lines per inch. Plate A is best adapted for general work, of which Fig. 35 is a type. For the purposes of railway location and construction, profiles are divided into Preliminary Location, Progress, and Final.

(*a*) Preliminary profiles are made for comparative purpose only. The approximate quantities of excavation and embankment are determined by scaling the distance from the grade line to the surface which corresponds to the centre height of the cut or fill; considering

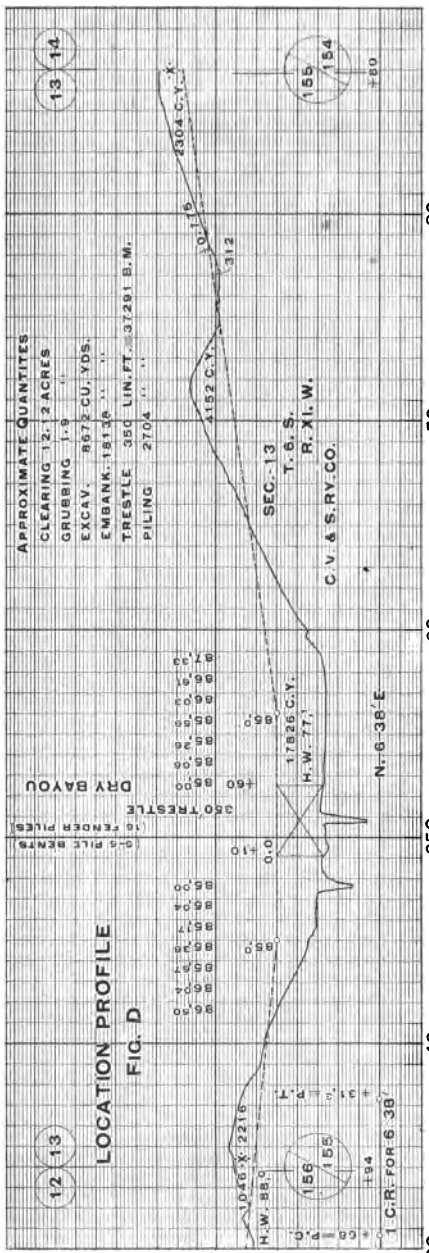
LOCATION PROFILE
FIG. D
APPROXIMATE QUANTITIES
CLEARING 12.2 ACRES
GRUBBING 1.9 "
EXCAV. 8672 CU. YDS.
EMBANK. 18136 " "
TRESTLE 350 LIN. FT.
PILING 2704 " "
37291 B.M.
DRY BAYOU
350 TRESTLE
SEC. 13
T. 6. S.
R. XI. W.
C. V. & S. RY. CO.
N. 6° 38' E.
1 C.R. = 6.38
H.W. 88.0
H.W. 77.1
1046 X 2216
1826 C.Y.
4152 C.Y.
2304 C.Y.
312
30
40
650
60
70
80

FIG. 35.

the section as level, the volume in cubic yards can be read off from the Earthwork Diagram. Estimates of clearing, grubbing, piling, trestling, etc., are also made. Of several preliminaries, the one best suited to all purposes as regards cost, maintenance, and operation, is selected, and along or near this the location is made.

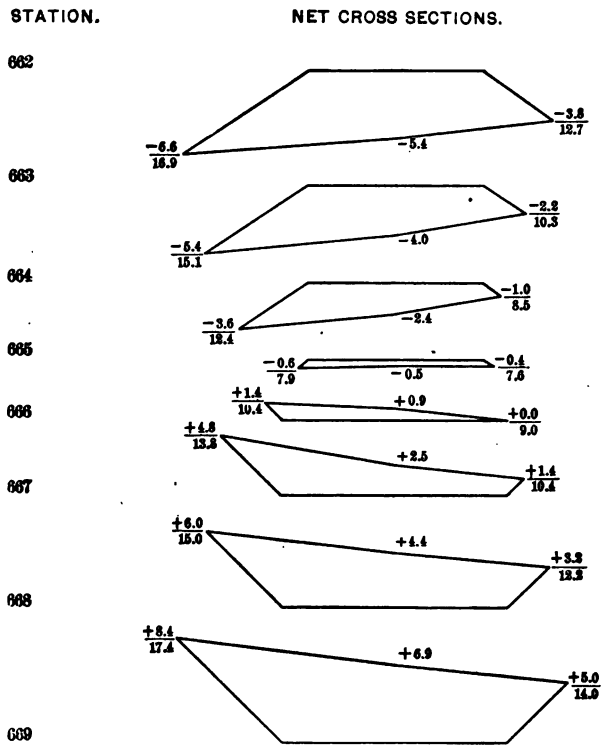

FIG. 36.

(*b*) *Location Profiles.*—After the location has been made, all curves run in, all grades, lengths of openings, size of culverts, pipes, etc., finally decided upon, the location profile is constructed, of which Fig. 35 is an example. As in case of preliminary profiles, the approximate amounts of excavation and embankment are estimated from the profile by calculation or by the aid of the Earthwork Diagram, and the volume in cubic yards recorded on each cut and fill. An estimate is also made of the piling, trestling, clearing, grubbing, etc., and all recorded in a general summary of approximate quantities for each section. The location profile is the "working" profile.

Each contractor should be furnished with a copy of his part of the work, each division engineer with a copy of that part which includes his division, and the resident and chief engineers should retain copies in their offices for reference. On it the alignment notes, high-water marks of the streams crossed, land-line ties, and ownership should also be recorded.

(*c*) *Progress Profile.*—The location profile can be used as a progress profile, but it is better to have a separate profile for this purpose. On it the work done in different months should be shown in different colors. The colors should be laid on lightly so as not to obscure the lines of the profile paper. A progress profile accompanies the monthly report and estimate of each division engineer, which record is duly transferred to the profiles of the resident and chief engineer.

(*d*) *Final Profile.*—After the completion of the work a final profile is prepared, which shows all data possible in connection with the construction of the road-bed and its condition at time of completion.

SECTION.

Station.	Alignment.	Surface.	Grade.	Coordinates.		
				Left.	Centre.	Right.
660		77.6	88.10	$-\frac{12.8}{27.2}$	− 10.5	$-\frac{8.6}{20.9}$
61		81.3	88.88	$-\frac{9.4}{22.1}$	− 7.6	$-\frac{6.4}{17.6}$
62		84.2	89.65	$-\frac{6.6}{16.9}$	− 5.4	$-\frac{3.8}{12.7}$
63		86.4	90.43	$-\frac{5.4}{15.1}$	− 4	$-\frac{2.2}{10.3}$
64		88.8	91.20	$-\frac{3.6}{12.4}$	− 2.4	$-\frac{1.0}{8.5}$
665		91.5	91.98	$-\frac{0.6}{7.9}$	− 0.5	$-\frac{0.4}{7.6}$
66		93.6	92.75	$+\frac{1.4}{10.4}$	+ 0.9	$\frac{0.0}{9.0}$
67		96	93.53	$+\frac{4.8}{13.8}$	+ 2.5	$+\frac{1.4}{10\ 4}$
68		98.7	94.30	$+\frac{6.0}{15.0}$	+ 4.4	$+\frac{3.2}{12.2}$
669		101.5	95.08	$+\frac{8.4}{17.4}$	+ 6.4	$+\frac{5.0}{14.0}$

FIG. 37.—PROPOSED

The actual grade line, alignment notes, position of all bridges, trestles, culverts, crossings, cattle-guards, etc., should be shown. The amount and classification of all quantities paid for during construction should be recorded on each section. A note of all land-line ties, ownerships of adjacent lands, width of right of way, records of deeds, etc., would add to its value. A copy of this profile should be retained in the office of the chief engineer as a reference.

61. Field-notes are permanently recorded in books specially ruled and marked for the purpose. Besides the ordinary transit and level books, the cross-sections of the road-bed at each station are plotted in a book (a sample page of which is shown in Fig. 36), and one similar to this can be used for sketching in topography. For a record of cross-section notes, amounts and classification of earthwork moved during construction, a field-book with pages ruled and marked as in Fig. 37, should be used. This is for the use of the division engineer and when filled out constitutes also a complete

Areas.			Cubic Yards.					Remarks.
			Excavation.			Embank.		
Earth.	Loose Rock.	Hard Rock.	Earth.	Loose Rock.	Hard Rock.	Earth.	Loose Rock.	

Form for Field-book.

record of all surface, grade, and alignment notes, and distribution of materials in excavation and embankment. In the "Remarks" column, notes on all clearing, grubbing, piling, trestling, and culverts, etc., can be recorded. This field-book, with the cross-section book (see illustration) and progress profile, constitutes the record of all construction work supervised by the division engineer, and on completion of the division the books and profiles are turned in by him to the office of the resident engineer.

The field-notes of an actual location are given in the field-book shown in Fig. 37. The classification is not given, as the data cannot be fully obtained till the grading is finished. One page of a cross-section book is given with the form of cross-section taken from the field-book in Fig. 37.

CHAPTER III.

CORRECTION FOR CURVATURE.

62. Let $E'ECDH$ be a railroad cut where $E'E$ is width of road-bed and $E'H$ and EC are the side slopes. For the sake of uniformity we shall, as usual, let all dimensions on left of centre line be indicated by primes. Thus $EE' = 2a$, $CT = h$, $ET = sh$, $AD = c$, $HQ = h'$, $E'Q = sh'$.

We shall indicate the coordinates at first base, mid-section, and

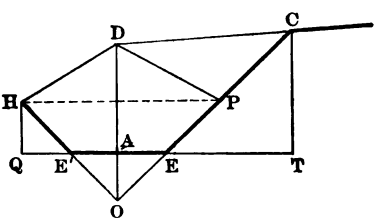

FIG. 38.

second base by subscripts. At first base the coordinates are h_1, sh_1, c_1, h', sh'; at mid-section $h_{\frac{1}{2}}$, $sh_{\frac{1}{2}}$, $c_{\frac{1}{2}}$, $h_{\frac{1}{2}}'$, $sh_{\frac{1}{2}}'$; and at second base h_2, sh_2, c_2, h_2', sh_2', and at any section a distance x from first base the coordinates are h_x, sh_x, c_x, h_x', sh_x', etc.

In terms of the general coordinates, the area (K) of $E'ECDH$ can be found as follows:

$$\text{Area } E'ECDH = AECD + AE'HD = ATCD - ECT + AQHD - E'QH$$
$$= \tfrac{1}{2}(AD + CT)AT - \tfrac{1}{2}CT \times ET + \tfrac{1}{2}(AD + HQ)AQ - \tfrac{1}{2}HQ \times E'Q$$
$$= (h + c)(a + sh) - \tfrac{1}{2}sh^2 + \tfrac{1}{2}(c + h')(a + sh') - \tfrac{1}{2}sh'^2$$
$$= ac + (a + cs)(h + h')$$
$$= \tfrac{1}{2}c(d + d') + \tfrac{1}{2}a(h + h') = K \quad . \quad . \quad .

Distance of c. of g. of $AE'H$ from $AD = \frac{1}{3}(2a + sh')$.

The areas of these triangles are, respectively,

$ADC = \frac{1}{2}c(a + sh)$, $ADH = \frac{1}{2}c(a + sh_1)$, $AEC = \frac{1}{2}ah$. $AE_1H = \frac{1}{2}ah'$.

By taking moments about centre of gravity, we get

$$\frac{c}{6}(a + sh)(a + sh - 3g) + \frac{a}{6}h(2a + sh - 3g)$$

$$= \frac{c}{6}(a + sh')(a + sh' + 3g) \frac{a}{6} h'(2a + sh' + 3g).$$

$$g = \frac{1}{6K}[2cs + s(h - h') + 2a^2 + sa(ch - h')](h - h'). \quad . \quad . \quad (D)$$

If we substitute in (C) and (D) the values of c, h, and h' at the section x we see that K is a quadratic function of x, and that g is a linear function of the same. Let the side slopes intersect at O. The triangle OEE' remains constant. $AO = a \div s$. Draw the line HP parallel to EE'.

$AE'HD = AEPD$. The triangle $DPC = AECD - ADHE'$.

$\therefore$ area $DPC = \frac{1}{2}[a(h+c)+shc-a(h'+c)-sh'c] = \frac{1}{2}(a+sc)(h-h')$.

Let g = distance of c. of g. of DPC from AD. By taking moments about OD and introducing OAE and $OE'A$ for simplicity, we have

$$\frac{g}{2}(a + sc)(h - h') = \frac{1}{6}\left(c + \frac{a}{s}\right)(a + sh)^2 - \frac{1}{6}\left(c + \frac{a}{s}\right)(a + sh')^2;$$

$$3g = 2a + s(h + h') = d + d'.$$

63. Let 13 in Fig. 39 be the centre line of a curved track and 2674

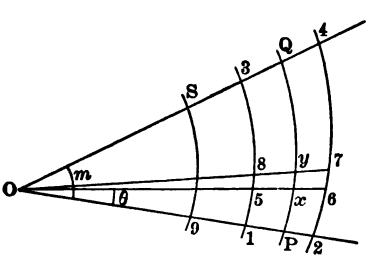

FIG. 39.

the intersection of the outer slope with the terrain, and 9S that of the inner slope.

Now c_1, c_2, and h_1, h_1', h_2, and h_2' are the centre cuts and side heights at stations (1) and (2) respectively. The two end sections 92

and $4S$ make an angle m at the centre O, and suppose 65 to be an intermediate section making an angle θ with first section. The centre cut and side heights at this intermediate section will be denominated respectively c_θ, h_θ, and h_θ'.

$$\therefore\ c_\theta = c_1 + \frac{\theta}{m}(c_2 - c_1);$$

$$h_\theta = h_1 + \frac{\theta}{m}(h_2 - h_1);$$

$$h_\theta' = h_1' + \frac{\theta}{m}(h_2' - h_1').$$

For convenience let $c_2 - c_1 = e$, $h_2 - h_1 = n$, $h_2' - h_1' = n'$, $h_1 - h_1' = f_1$, $h_2 - h_2' = f_2$.

If K_θ represents the area of triangle DPC in section on OC, we obtain

$$K_\theta = \tfrac{1}{2}(a + sc_\theta)(h_\theta - h_\theta') = \tfrac{1}{2}(a + sc_1 + \frac{s\theta}{m}l)(f_1 + \frac{\theta}{m}(n - n'))$$

$$= \tfrac{1}{2}f_1(a + sc_1) + \frac{\theta}{2m}[sef_1 + (n - n')(a + sc_1)] + \frac{se\theta^2}{m^2}(n - n').$$

$$3g_\theta = 2a + s(h_\theta + h_\theta') = 2a + h_1 + h_1' + \frac{s\theta}{m}(n + n').$$

In the same way we can establish the fact that any other area in the cross-section that has variable coordinates is a quadratic function of $\theta \div m$, and that the distance of its centre of gravity from the central line is a linear function of $\theta \div m$. The triangles AEC and AEH are linear functions of $\theta \div m$, as they rest on a constant base. If l = distance 1583, then $Rm = l$.

64. General Proof of Correction Formulæ.—From the theorem of Pappus (or Guldin) we know that the volume generated by an area revolving around an axis in the plane of its area is equal to the area multiplied by the path of the centre of gravity.

Let PQ, Fig. 39, be the path of the centre of gravity of the cross-section and xy = an infinitesimal length of path of $cg = ds$.

Its normal component $= ox \times d\theta = (R + g)d\theta$.

We may represent K_θ and g_θ under general form as follows:

$$K_\theta = A + B\frac{\theta}{m} + C\frac{\theta^2}{m^2}; \quad \ldots\ldots\ (E)$$

$$g = D + E\frac{\theta}{m}. \quad \ldots\ldots\ (F)$$

But $dV = K_\theta(R + g)d\theta\left(A = B\frac{\theta}{m} + C\frac{\theta^2}{m^2}\right)\left(R + D + E\frac{\theta}{m}\right)d\theta.$

$$\therefore V = \int_0^m \left[\left(A + B\frac{\theta}{m} + C\frac{\theta^2}{m^2}\right)\left(R + D\right) + E\left(A\frac{\theta}{m} + B\frac{\theta^2}{m^2} + C\frac{\theta^3}{m^3}\right)\right] d\theta$$

$$= Rm\left(A + \frac{B}{2} + \frac{C}{3}\right) + Dm\left(A + \frac{B}{2} + \frac{C}{3}\right) + Em\left(\frac{A}{2} + \frac{B}{3} + \frac{C}{4}\right)$$

$$= l\left(A + \frac{B}{2} + \frac{C}{3}\right) + \frac{Dl}{R}\left(A + \frac{B}{2} + \frac{C}{3}\right) + \frac{El}{R}\left(\frac{A}{2} + \frac{B}{3} + \frac{C}{4}\right). \quad \text{(G)}$$

In formulæ (E) and (F) make $\theta = 0$, $\frac{1}{4}m$, $\frac{3}{4}m$, and m.

$$\therefore K_1 = A, \qquad g_1 = D,$$

$$K_{\frac{1}{4}} = A + \frac{B}{2} + \frac{C}{4}, \qquad g_{\frac{1}{4}} = D + \tfrac{1}{4}E,$$

$$K_2 = A + B + C, \qquad g_2 = D + E,$$

$$K_{\frac{3}{4}} = A + \frac{2B}{3} + \frac{4C}{9}, \qquad g_{\frac{3}{4}} = D + \frac{2E}{3}.$$

Also, $K + 2K_{\frac{1}{4}} = 3A + B + \frac{C}{2}$, $\qquad K_2 + 2K_{\frac{1}{4}} = 3A + 2B + \frac{3C}{2}$,

$$g_1 + g_2 = 2g_{\frac{1}{4}}.$$

If the area K_θ should move simply by translation, its general form would be

$$K_x = A + Bx + Cx^2.$$

The volume generated by K_x in distance l is given in Article 19. Calling this volume V_s, we have

$$V_s = \left(A + \frac{B}{2} + \frac{C}{3}\right)l.$$

The error or correction for curvature is therefore represented by the last two terms of (G).

$$\text{Correction} = \frac{l}{R}\left[D\left(A + \frac{B}{2} + \frac{C}{3}\right) + E\left(\frac{A}{2} + \frac{B}{3} + \frac{C}{4}\right)\right] \quad \text{(H)}$$

$$= \frac{l}{6R}[K_1 g_1 + 4K_{\frac{1}{4}} g_{\frac{1}{4}} + K_2 g_2] \quad \text{(16)}$$

$$= \frac{l}{6R}[(K_1 + 2K_{\frac{1}{4}})g_1 + (K_2 + 2K_{\frac{1}{4}})g_2]$$

$$= \frac{l}{6R}[K_1 g_1 + 2K_{\frac{1}{4}}(g_1 + g_2) + K_2 g_2]. \quad \text{(I)}$$

Formulæ (16) and (I) can be easily remembered by their similarity to two well-known formulæ. Formula (16) is of the same form as

the Newtonian prismoidal formula, while (I) is analogous to the "three-moment" formula of higher mechanics.

Formula (H) can assume the following form:

$$\text{Correction} = \frac{l}{4R}[AD + 3(A + \tfrac{1}{3}B + \tfrac{1}{3}C)(D + \tfrac{1}{3}E)] + \frac{l}{36R}EC$$

$$= \frac{l}{4R}[K_1 g_1 + 3K_{\frac{2}{3}} g_{\frac{2}{3}}] + \frac{l}{36R}EC.$$

If $C = 0$, we have a two-term correction formula.

If K_θ is a cubic function of $\frac{\theta}{m}$, that is, if

$$K_\theta = A + B\frac{\theta}{m} + C\frac{\theta^2}{m^2} + F\frac{\theta^3}{m^3},$$

and $$= \qquad g_\theta = D + E\frac{\theta}{m},$$

the correction is given by the following formula:

$$\text{Correction} = \frac{l}{6R}[K_1 g_1 + 4K_{\frac{1}{2}} g_{\frac{1}{2}} + K_0 g_0] - \frac{l}{120R}(EF).$$

PROBLEM 8.—The road at a crest is graded to the form of the common parabola each way where $Y = \frac{1}{10000}x^2$. The surface ground longitudinally was falling at the rate of 7 to 100, and transversely at the rate of 1 in 6 toward centre of curve. The centre cuts are given by

$$Cx = 24 + \tfrac{1}{10000}x^2 - \tfrac{7}{100}x.$$

Therefore $h_x = \frac{1}{3}(c + 2)$, $h_x' = \frac{1}{3}(c - 2)$, where $a = 12$ and $s = \frac{2}{3}$.

Area of DPC (see Fig. 38), $K_x = \frac{2}{3}(c_x + 8)^2$, $g_x = 8 + \frac{1}{4}\frac{6}{8}(2c_x + 1)$.

If $R = 2000$, find correction in volume due to curvature.

65. Rankine's Method of Correction.—"Where earthwork occurs on a sharp curve. Calculate ET and $E'Q$ (see Fig. 38) required for the two slopes; take their difference and divide it by three times the radius of the curve: the quotient is to be added or subtracted from unity according as the greater half breadth lies from or towards the centre of the curve. The result will be a factor by which the area HOC in Fig. 38 is to be multiplied. From the product subtract the area OEE'; the remainder will be the area modified for curvature; then proceed usual."

In Fig. 38 the quadrilateral $OHPD$ has its centre of gravity on the centre line DA, and the volume generated by it will need no correction. The correction must, therefore, be made for the triangle DPC.

Let g = distance of centre of gravity of *OHDC* from *OD*. By taking moments about the centre of gravity we have

$$\tfrac{1}{2}(a+sh'+3g)\left(c+\frac{a}{s}\right)(a+sh') = \tfrac{1}{2}(a+sh-3g)\left(c+\frac{a}{s}\right)(a+sh).$$

$$\therefore g = \tfrac{1}{6}(sh-sh').$$

$$\text{Area } OHDC = K = \frac{1}{2}\left(c+\frac{a}{s}\right)(2a+sh+sh').$$

$$\therefore dv = \left(c+\frac{a}{s}\right)(2a+sh+sh')(R+g)d\theta.$$

$$\therefore V = \frac{l}{2}\left(c+\frac{a}{s}\right)(2a+sh+sh') + \frac{l}{6R}\left(c+\frac{a}{s}\right)(2a+sh+sh')(sh-sh')$$

$$= \frac{l}{2}\left(c+\frac{a}{s}\right)(2a+sh+sh')\left[1+\frac{sh-sh'}{3R}\right]$$

$$= lK\left(1+\frac{d-d'}{3R}\right) = lK\left(1+\frac{sh-sh'}{3R}\right) = lK\left(1+\frac{d-d'}{3R}\right). \quad (17)$$

If the ground is higher on concave side, $(sh-sh')$ becomes negative; hence the rule as stated. This correction is based on the hypothesis that the area remains constant throughout the length.

Formula (17) gives the correction of the volume generated by the whole area *OHDC*. We must subtract the volume generated by *OEE'*. Calling this area K', we get exact volume generated by *ECDHE'*

$$= l\left[K\left(1+\frac{sh-sh'}{3R}\right) - K'\right]. \quad \ldots\ldots \quad (J)$$

This formula (J) was published in John Warner's "Earthwork" (1861), and Rankine adapted and simplified it as above.

PROBLEM 9. Given $2a = 28$, $h = 40$, $h' = 16$, $c = 28$, $s = \frac{3}{2}$, $R = 1400$, $l = 100$, find exact volume and correction due to curvature.

Volume = 197792 cu. ft. Correction = 1792 cu. ft.

66. Henck's Method of Correction.—Let 789 be the centre line, o the centre of curve, *A*23 intersection of inner side slope with terrain, and 654 that of outer.

The sections normal to curve at 7, 8, and 9 are 16, 25, and 34; but if the sections are taken perpendicular to the chords 78 and 89, these will be *BA*, *CD*, *FE*, and *GH*.

The volume between 7 and 8 will be that included between 16 and 25. This is equal to that included between *AB* and *FE* if the generating area is constant.

If we use the sections perpendicular to the chords in estimating the volumes, we take in *D*8*E* too much and leave out that of *F*8*C*.

At 7 we take in 17*A* and omit 76*B*. If the ground is level, these areas and volumes practically balance each other, for the short arcs *DE* and *FC* can be taken as straight, and on level ground 85 = 28.

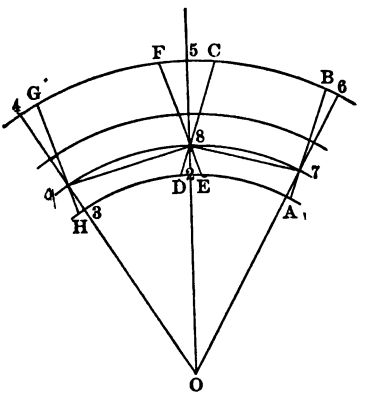

FIG. 40.

The mid-ordinates of *DE* and *FC* will be calculated for the special example given by Henck.

$$\text{Mid-ordinate of } FC = \frac{(FC)^2}{8(R+d)}.$$

$$\text{Mid-ordinate of } DE = \frac{(DE)^2}{8(R-d')}.$$

Take $R = 1400$, arc $78 =$ arc $89 = 100'$, angle $405 = 4^\circ\ 5' =$ angle $F8C =$ degree of curve, $h = 40$, $h' = 16$, $2a = 28$, $s = \frac{3}{8}$.

$$DE = \frac{38}{57.3} \times 4\tfrac{1}{16} = 2.72 \text{ ft.}$$

$$FC = \frac{74}{57.3} \times 4\tfrac{1}{16} = 5.27 \text{ ft.}$$

$$\text{Mid-ordinate } FC = \frac{(5.27)^2}{8 \times 1472} = .48 \text{ inch;}$$

$$\text{Mid-ordinate } DE = \frac{(2.72)^2}{8 \times 1362} = .00814 \text{ inch.}$$

It is evident from this last investigation that we can, with entire confidence as to the accuracy of our work, let the volume *DE*8 and *FC*8 balance each other on a level. It is, however, unnecessary to enter into these considerations when the side heights are equal, for the centre of gravity of the cross-section is always on the centre line.

From Fig. 38 we see that all correction must be due to the triangle

DPC, which is known as the correction triangle. Henck treats the volume generated by this triangle in moving from $8C$ to $8F$ as a prism. However, instead of following Henck's Method in using the sines of the degree of curve, we can use the circular measure, as in this case it is more accurate and is always shorter. We have always

$$D = \frac{5730}{R}.$$

$$FC = \frac{78C}{57{\cdot}3} \times 8C = \frac{D}{57{\cdot}3}(a + sh) = \frac{100}{R}(a + sh) = \frac{100d}{R};$$

$$DE = \frac{D}{57{\cdot}3} \times (a + sh') = \frac{100}{R}(a + sh') = \frac{100d'}{R}.$$

Volume generated by *DPC* is equal to the base multiplied by the one-third of the sum of the three edges.

$$\text{Area base} = \tfrac{1}{2}(a + cs)(h - h') = K.$$

$$\text{Average height} = \frac{100}{3R}(2a + sh + sh') = \frac{100}{3R}(d + d').$$

$$\text{Volume correction} = \frac{100}{6R}(a + cs)(h - h')(d + d'). \quad . \quad . \quad . \quad (18)$$

In the example we have, correction $= \frac{100}{6 \times 1400} \times 56 \times 112 = 1792$ cu. ft.

Another method based on Henck's: The distance of centre of gravity of *DPC* from centre line is, from Art. 62, $= \frac{1}{3}(2a + sh + sh') = \frac{1}{3}(d + d')$. Multiply the area *DPC* by the path of its cg, and we obtain the volume.

$$\text{Path of } cg = \frac{D}{57{\cdot}3}(2a + sh + sh')\tfrac{1}{3} = \frac{100}{3R}(2a + sh + sh') = \frac{100}{3R}(d + d');$$

$$\text{Volume} = \tfrac{1}{2}(a + cs)(h - h') \times \frac{100}{3R}(2a + sh + sh')$$

$$= \frac{100}{6R}(a + cs)(h - h')(2a + sh + sh')$$

$$= \frac{100}{6R}(a + cs)(h - h')(d + d') = \frac{100K}{3R}(d + d'),$$

$$= \frac{100Kg}{K} = \frac{100KgD^{\circ}}{5730}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

This formula is by far the shortest and simplest yet determined. Its applicability to cases where the volume is not constant will be noticed later.

67. If we wish to find the correction in cubic yards, we have

$$C = \frac{100}{81 \times 5730} \times KD(d + d') = \frac{100}{81 \times 5730} KWD = \frac{KWD}{4640}, \quad . \quad (20)$$

where $W = d + d'$.

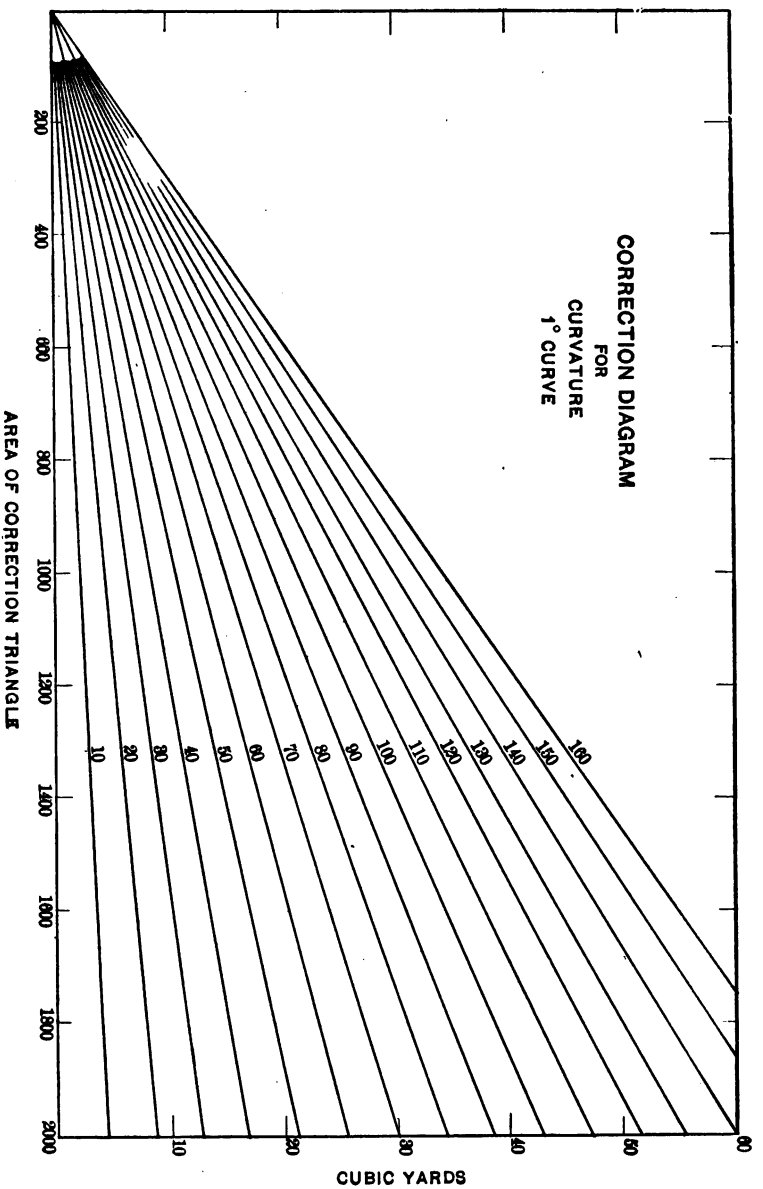
CORRECTION DIAGRAM
FOR
CURVATURE
1° CURVE
AREA OF CORRECTION TRIANGLE
200
400
600
800
1000
1200
1400
1600
1800
2000
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160
10
20
30
40
50
60
CUBIC YARDS

FIG. 41.

The correction therefore varies directly with the degree of curvature, and we can assume a one-degree curve and find corrections for various values of K and W, and then multiply by the degree of curvature.

$$\therefore\ C = \frac{KW}{4640}.$$

If we assume W equal to (say) 20, then C varies directly as K, and the equation represents a straight line. Assume $K = 1160$ and $W = 20$. $\therefore\ C = 5$. By joining the points whose coordinates are (0, 0) and (1160, 5) we locate the 20-line. In the same way we can locate the 20-line, the 30-line, the 40-line, etc. **Example**: To locate the 40-line we assume $W = 40$ and $K = 1160$. $\therefore\ C = 10$. Lay off 1160 on horizontal line from 0 and erect a perpendicular at the point thus found, and lay off to any scale 10 parts on this perpendicular measuring from the horizontal line. Join the last point determined with 0. This is the 40-line. Then after the diagram (Fig. 41) is constructed, to find the correction for any width and area, we find on the horizontal line the area, and follow the vertical to the line representing the width; then read off the correction in cubic yards in the right margin. This will be the correction on a one-degree curve; and to find the correction on any other curve we have simply to multiply by the degree of curve.

In the problem given above we find

$$K = (a + cs)(h - h') = \tfrac{1}{2}(12 + 28 \times \tfrac{3}{2})(40 - 16) = 672 \text{ sq. ft.}$$

$$g = \tfrac{1}{3}(2a + sh + sh') = \tfrac{1}{3}(28 + 60 + 24) = 37\tfrac{1}{3} \text{ ft.}$$

$$\text{Volume} = \frac{100}{R}Kg = \frac{100 \times 672}{1400} \times 112 = 1792 \text{ cu. ft.}$$

68. General Formula for Area.—From Art. 62, we have

$$K = (a + sc)(h - h'),$$

$$3g = 2a + s(h + h') = d + d';$$

but for any intermediate section obtain

$$c_\theta = c_1 + \frac{\theta}{m}(c_2 - c_1);$$

$$h_\theta = h_1 + \frac{\theta}{m}(h_2 - h_1)$$

$$h'_\theta = h'_1 + \frac{\theta}{m}(h'_2 - h_1').$$

$$K_\theta = \tfrac{1}{2}(a+sc_\theta)(h_\theta - h_\theta') = \tfrac{1}{2}(a-sc_1)(h_1-h_1') + \frac{\theta}{2m}(a+sc_1)(h_2-h_1+h_1'-h_2')$$
$$+ s(h_1-h_1')(c_2-c_1) + \frac{s\theta^2}{m^2}(c_2-c_1)(h_2-h_1+h_1'-h_2')$$
$$= K_1 + \frac{\theta}{2m}[(a+sc_1)(h_2-h_1+h_1'-h_2') + s(h_1-h_1')(c_2-c_1)]$$
$$+ \frac{s\theta^2}{m^2}(c_2-c_1)(h_2-h_1+h_1'-h_2'). \quad . \quad . \quad . \quad . \quad . \quad (K)$$

$$3g_\theta = 2a + s(h_\theta + h_\theta') = 2a + s(h_1+h_1') + \frac{s\theta}{m}(h_2-h_1+h_2'-h_1')$$
$$= 3g + \frac{s\theta}{m}(h_2-h_1+h_2'-h_1'). \quad . \quad . \quad . \quad (L)$$

It is unnecessary to carry out the proof for correction formulæ for (K) and (L), as we have already established this formula for a general quadratic equation; and, as all ordinary earthwork cuts fall under the form of Fig. 38, the general law as indicated in formulæ (H), (16), (I), applies.

Example.—Given $2a = 28$, $h_1 = 44$, $h_1' = 22$, $c_1 = 30$, $h_2 = 36$, $h_2' = 10$, $c_2 = 26$, $R = 1400$, $l = 200$, find total correction.

$$K_1 = 644, \quad K_2 = 344.5, \quad K_{\frac{1}{2}} = 672,$$
$$g = \tfrac{1}{3}(127), \quad g_2 = \tfrac{1}{3}(97), \quad g_{\frac{1}{2}} = \tfrac{1}{3}(112).$$
$$\text{Correction} = 3574 \text{ cu. ft.}$$

If we assume that the area remains constant while describing the small arcs $B6$, CF, and $G4$, Fig. 40, we can apply formula (19) for each section.

(1) Correction at Station (7).

$$K_1 = \tfrac{1}{2}(a + sc)(h - h') = \tfrac{1}{2}(14 + 45)(22) = 649;$$
$$3g = 2a + s(h_1 + h_1') = 127.$$
$$\text{Correction} = \frac{50Kg}{R} = 981.22 \text{ cu. ft.}$$

(2) Correction at station (9).

$$K_2 = 689, \quad 3g_2 = 97.$$
$$\text{Correction} = \frac{50Kg}{R} = 795.63.$$

(3) Correction at station (8).

$$K_{\frac{1}{2}} = 672, \quad 3g_{\frac{1}{2}} = 112.$$

$$\text{Correction} = \frac{100Kg}{R} = 1792.$$

Total approx. correction = 3568.85 cu. ft.

Exact correction = 3574.00 cu. ft.

Error or approx. correction = 5.15 cu. ft. = .57 cu. yd.

PROBLEM 11.—Given in a two-level section $h_1 = 24$, $h_1' = 12$, $2a = 20$, $h_2 = 10$, $h_1' = 30$, on a six-degree curve, find exact and approximate corrections in cubic yards.

PROBLEM 12.—In the following table $s = \frac{2}{3}$, $2a = 18$, $l = 100$, $R = 1500$, fill out volume and correction columns.

Sta.	h	C	h'	K	g	Volume.		Correction for Curvature.		
						By Pris. Form.	By Av. End Areas.	Exact.	Read from Fig. 37.	Rankine.
0	8	4	2							
1	20	14	10							
2	30	22	16							
3	42	30	24							
4	30	24	20							
5	20	18	14							
6	16	16	16							

PROBLEM 13.—Find correction for curvature in Problem 8 for an eight-degree curve.

PROBLEM 14.—The notes at station (1) are $[\frac{18}{18}, 14, \frac{16}{16}]$, and those at station (2) are $[\frac{18}{17}, 16, \frac{10}{16}]$. Find correction for curvature on a six-degree curve.

PROBLEM 15.—Fill out field-book in Fig. 37.

CHAPTER IV.

TWO-TERM FORMULÆ.

69. To find an expression for the mean area of a prismoid in terms of the bases and another section.

It is thought that the following method of deriving such a prismoid formula is published for the first time.

Let B_1, B_2 be respectively the lower and upper bases of a prismoid.

Let S be any section parallel to B_1 and B_2;

H = altitude of prismoid B_1B_2;
a = altitude of prismoid B_1S;
c = altitude of prismoid B_2S;
A_a = base of associate cone for prismoid B_1S;
A_c = base of associate cone for prismoid B_2S;
A = base of associate cone for prismoid B_1B_2;
M = mean area of prismoid;
$x = a \div H$;
$z = c \div H$.

$$\text{Vol. } B_1S = \left(\frac{B_1 + S}{2} - \frac{1}{6}A_a\right)a = \left(\frac{B_1 + S}{2} - \frac{a^2}{6H^2}A\right)a;$$

$$\text{Vol. } B_2S = \left(\frac{B_2 + S}{2} - \frac{1}{6}A_c\right)c = \left(\frac{B_2 + S}{2} - \frac{c^2}{6H^2}A\right)c.$$

Adding we get the total volume.

$$\therefore V = \frac{S(a + c)}{2} + \frac{aB_1}{2} + \frac{cB_2}{2} - \frac{a^2 + c^2}{6H}A.$$

But $V = MH$. $\quad \therefore M = V \div H.$

$$\therefore M = \frac{S}{2} + \frac{xB_1}{2} + \frac{yB_2}{2} - \frac{x^2 + z^2}{6}A. \quad \ldots\ldots \text{(M)}$$

But $M = \dfrac{B_1 + B_2}{1} - \dfrac{1}{6}A.$

Substitute the value of A found from (3) in (M), and we get

$$M = \frac{B(x - x^3 - z^3)}{2(1 - x^3 - z^3)} + \frac{S}{2(1 - x^3 - z^3)} + \frac{B(y - x^3 - z^3)}{2(1 - x^3 - z^3)}. \quad (N)$$

Now $x + z = 1$. $\therefore x^3 + z^3 = 1 - 3xz(x + z) = 1 - 3xz$.

$$\therefore M = \frac{B_1(3x - 1)}{6x} + \frac{S}{6xz} + \frac{B_2(3z - 1)}{6z} \quad (21)$$

$$= B_1\frac{(3x - 1)}{6x} + \frac{S}{6x(1 - x)} + \frac{B_2(2 - 3x)}{6(1 - x)} \quad (22)$$

$$= fB_1 + eS + dB_2,$$

where f, e, d represent the coefficient of B_1, S, and B_2 respectively.

$$\therefore f = \frac{3x - 1}{6x}, \quad e = \frac{1}{6x(1 - x)}, \quad d = \frac{2 - 3x}{6(1 - dx)}.$$

Transfer the origin for f to $(0, \frac{1}{2})$ and for d to $(1, \frac{1}{2})$, and we get

$$fx = \tfrac{1}{6}, \quad dx = \tfrac{1}{6},$$

which are the equations to equilateral hyperbolas referred to asymptotes OB and EF, and PQ and EF (Fig. 42).

If for e we transfer origin to $(\frac{1}{2}, 0)$, we get

$$e = \frac{2}{3 - 12x^2}. \quad (O)$$

70. By making x vary from 0 to 1, we can get the following table of coefficients :

x.	f.	e.	d.
0			
.05	− 2.833	3.508	.325
.10	− 1.167	1.350	.315
.15	− .611	1.307	.304
.20	− .333	1.042	.292
.2114	− .2886	1.00	.2886
.25	− .1666	.888	.277
.30	− .0555	.7936	.262
$\frac{1}{4}$	0	.7500	.250
.35	.00238	.732	.243
.40	.0833	.6946	.2222
.45	.129	.673	.1969
.50	.167	.6666	.1667
.55	.197	.6730	.130
.60	.222	.6946	.083
.667	.250	.7500	.00
.70	.262	.7936	.0555

By using x as abscissas and the corresponding coefficients as ordinates, we get the "coefficient" curves as shown in Fig. 6 It is noticed that the "coefficient" of S is symmetrical with reference to the line $K1$, and that the curves (f) and d) are symmetrical, with respect to each other, with reference to the same line.

The (e) curve is asymptotic to lines $x = 0(OB)$ and $x = 1(PQ)$. The (f) curve is asymptotic to OB and line $a = \frac{1}{2}(FE)$.

It can be seen from equation (O) that the (e) curve is asymptotic to lines $x = \pm \frac{1}{2}$.

71. Coefficient Curves for Three-term Formulæ.—From Fig. 42, where OP = unity, we readily see that by taking values of x from 0 to 1 the ordinate to (d) curve gives the coefficients of B_2, the ordinate to (e) curve gives the coefficients of the floating section S, and the ordinate to the (f) curve gives the coefficient of B_1. When

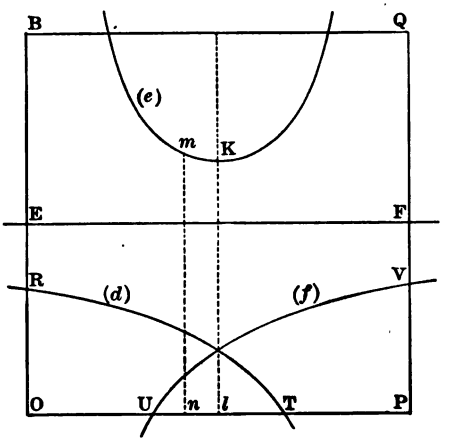

FIG. 42.

these sections are multiplied by their coefficients and added, a *mean area* is obtained, which, multiplied by the height of the prismoid, will give the true volume.

When $x = \frac{1}{4}$, the coefficient of B_1 disappears, and we get

$$M = \tfrac{1}{4}B_2 + \frac{3S}{4},$$

thus verifying by the three-term formula Kinklin's two-term formula.

When $x = \frac{1}{3}$, we get

$$M = \tfrac{1}{4}B_1 + \frac{3S}{4},$$

another verification of the same formula. This two-term formula can be considered a special case of three-term formula. By inspecting Fig. 6 or the table I, we see that if we wish *positive* coefficients we must take S in the *middle third* of the height, and, as it leaves the *middle third*, it verifies the two-term formula.

The coefficients are symmetrical with reference to Kl or the mid-section.

When $x = \frac{1}{2}$, we get

$$M = \tfrac{1}{6}(B_1 + B_2 + 4S_{\frac{1}{2}}).$$

72. By the use of equations (3) and (22) a two-term prismoidal formula can be derived by elementary geometry as follows:

From (22) we have

$$M = B_1\frac{3x-1}{6x} + \frac{S_2}{6x(1-x)} + B_2\frac{2-3x}{6(1-x)};$$

then for any other section (S_1) distant yH from B_1, we have

$$M = B_1\frac{3y-1}{6y} + \frac{S_1}{6y(1-y)} + B_2\frac{2-3y}{6(1-y)}. \quad . \quad . \quad . \quad . \quad (23)$$

But $$B_2 = 2M - B_1 + \frac{A}{3}.$$

By substituting this in (22) and (23), and reducing, we get

$$M = B_1\frac{2x-1}{2x} + \frac{S_2}{2x} + \frac{A(2-3x)}{6},$$

$$M = B_1\frac{2y-1}{2y} + \frac{S_1}{2y} + \frac{A(2-3y)}{6}.$$

Eliminate B_1.

$$\therefore M = S_2\frac{2y-1}{2(y-x)} + \frac{S_1(1-2x)}{2(y-x)} + \frac{6xy-3y-3x+2}{6}A. \quad (24)$$

Now if the coefficient of A in (24) disappears, we get a two-term prismoidal formula.

The condition for two-term formulæ is, therefore,

$$6xy - 3y - 3x + 2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

By transferring the origin to ($\frac{1}{2}$, $\frac{1}{2}$), we get

$$xy = -\tfrac{1}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (P)$$

It is readily seen that this is an equilateral hyperbola referred to its asymptotes.

The coefficients of S_2 and S_1 are

$$p = \frac{2y-1}{2(y-x)} = \frac{1}{4(3x^2-3x+1)}, \quad q = \frac{1-2x}{2(y-x)} = \frac{3(1-2x)^2}{4(3x^2-3x+1)}. \quad (Q)$$

Transfer the origin to ($\frac{1}{2}$, 0).

$$\therefore p = \frac{1}{1+12x^2}, \quad q = \frac{12x^2}{1+12x^2}. \quad . \quad . \quad . \quad . \quad (R)$$

If $OR = OQ = H$, say 100, and OA and OC represent the respective axes, the branches of the hyperbola can be plotted, and the ordinates and abscissas to the branch 123 between 7 and 8 and 456 between 4 and 6 will locate all the sections between the bases. (Fig. 43.)

The "coefficient" curve is obtained from equations (R), where for any value of x the ordinates from OA to the coefficient curve will give the coefficient of S_2 and the ordinates down from QD to the same point on the coefficient curve will give the coefficient of S_1.

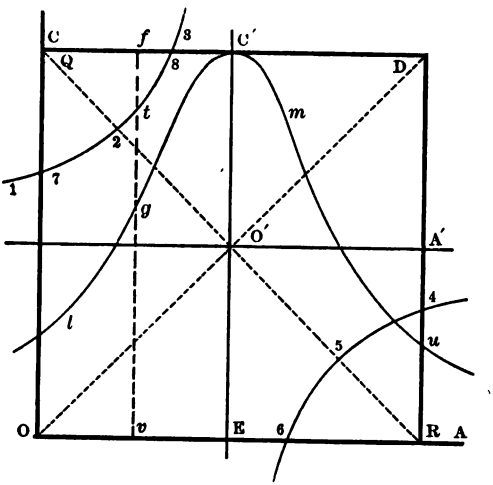

FIG. 43.

73. By easy calculation the following values of x, y, p, and q can be found:

x	y	Coefficients. Of S_1. p	Coefficients. Of S_2. p	h
.00	.666	.750	.250	.6667
.05	.685	.709	.291	.635
.10	.709	.659	.443	.609
.15	.738	.595	.405	.5880
.20	.778	.519	.481	.5780
.21½	.7886	.500	.500	.5776
.25	.833	.429	.571	.5833
.30	.917	.324	.676	.617
.33⅓	1.000	.250	.750	.6667
.35	1.056	.212	.788	.756
.37½	1.162	.158	.842	.7916
.40	1.333	.108	.892	.9333
.43	1.693	.0556	.9444	1.063
.45	2.167	.029	.971	1.717
.47	3 278	.011	.989	2.808
.48	4.662	.00481	.995	4.186
.49	8.833	.002	.998	8.343
.50	Infinity	.0	1 000	Infinity
.60	−.33⅓	.108	.892	
.667	.0	.250	.750	
.70	.083	.324	.676	
.80	.222	.519	.481	
.90	.292	.657	.343	
1.00	.333	.750	.250	

The coordinates of 7, 8, 6, and 4 locate the sections for Kinklin's two-term formulæ, and the ordinates to the "coefficient" curve give $\frac{3}{4}$ and $\frac{1}{4}$ for S_1 and S_2 and *vice versa*.

The coordinates of 2 and 5 locate the two sections equidistant from the bases, where $x = .2114$ and $y = .7886$, as given by Prof. Echols, and the "coefficient" curve crosses the asymptote on same ordinate, and gives $p = .50$ and $q = .50$. The graphical construction shows that no real section of the prismoid (between bases) can occur for values of x between 8 and 6; i.e., within the middle third. However, good results can be obtained for all values of x from 0 to .45 and from .55 to 100. No sections can be taken at or very near the centre. *Example:* For any position, $ov = x$, $vt = y$, $fg = q$, and $vg = p$.

74. "Position" and "Coefficient" Curves.—(1) To find the volume of any prismoid measure one section (S_1) at a distance $xH = ov$ in Fig. 43, and measure the other section (S_2) at a distance from same base $yH =$ the ordinate to the "position" curve $728 = vt$ for the abscissa vo. Then multiply the first section by vg and the second section by fg, where $H = OQ = OR =$ unity.

(2) The "position" curves are asymptotic to the lines $y = \frac{1}{2}$ and $x = \frac{1}{2}$, or to OA and OC.

(3) The vertices of the "position" curves (hyperbolas) are on the diagonal QR of the square $ORDQ$.

(4) The point of inflection of the "coefficient "curve is vertically below (above) the points 8 and 6.

(5) The "coefficient" curve crosses asymptote $O'A'$ on an ordinate through vertex of "position" curve.

(6) For the last position, S_2 is at a distance $.2114H$ from one base and S_1 the same distance from the other base, and the coefficients here are each $\frac{1}{2}$.

(7) When $x = 0$, $y = .667$, and the coefficients are .250 and .750 respectively, thus again verifying Kinklin's formula.

(8) When $x = .333$, $y = 1$, and the coefficients are .750 and .250 respectively.

(9) At the points $x = 0$, .333, .667, or 1.00, the "position" curve leaves the square, and the branches 728 and 456 give all the sections between the bases.

(10) The diagram (Fig. 43) and table show very plainly that no real section (between bases) *can be taken in the middle third.* The three-term formulæ for *positive* coefficients requires the section to be taken *in the middle third.*

75. In the *Annals of Mathematics* for November, 1894, Prof. Echols has shown that there is an infinite number of two-term formulas that will give the mean area of the prismoid. It is the object of the present article to follow his general method in order to verify the calculations of the preceding article.

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For convenience the two parallel sections will be taken between bases B_2 and B_1. The section next to B_2 will be called S_1, and that next to B_1, S_2. Then, let P be any section, and the

Distance from P to section S_2 $= m$,
Distance between S_1 and S_2 $= h$,
Distance from P to section S_1 $= h - m$,
Distance between section B_1 and S_2 $= a$,
Distance between bases B_1 and B_2 $= H$,
Distance between P and base B_1 $= z$.

From Elliott's extension of Holditch's theorem we have

$$P = \frac{mS_1}{h} + \frac{(h-m)S_2}{h} - \frac{m(h-m)A_h}{h},$$

where A_h = area of base of "associate cone" between sections S_1 and S_2. It is clear that

$$m = z - a;$$

$$\therefore \quad P = \frac{(z-a)S_1}{h} + \frac{(h-z+a)S_2}{h} - \frac{(z-a)(h-z+a)A_h}{h}. \qquad (S)$$

Now we wish to find the mean area of the section P between sections B_1 and B_1 in terms of S_1, S_2, and A_h. Using the well-known formula for mean value for $y = F(x)$ between the limits of a and b for x, we have

$$M = \text{mean value} = \frac{1}{a-b}\int_b^a F(x)dx.$$

From (S), then,

$$M = \frac{1}{H-0}\int_0^H \left[\left(\frac{z-a}{h}\right)S_1 + \frac{h-z+a}{h}S_2 - \frac{(z-a)(h-z+a)}{H^2}A\right]dz$$

$$= \frac{H-2a}{2h}S_1 + \frac{2h-H+2a}{2h}S_2 - \frac{3hH-2H^2+6aH-6ah-6a^2}{6H^2}A.$$

Now make $h = c - a$, where c = distance S_1B_1.

$$\therefore \quad M = \frac{H-2a}{2(c-a)}S_1 + \frac{2c-H}{2(c-a)}S_2 - \frac{3Hc+3aH-6ac-2H^2}{6H^2}A.$$

Make $a = xH$ and $c = yH$.

$$\therefore \quad M = \frac{2y-1}{2(y-x)}S_2 + \frac{1-2x}{2(y-x)}S_1 + \frac{6xy-3x-3y+2}{6}A.$$

Now, as in (24), for a two-term formula, the coefficient of A must disappear;

$$\therefore \quad 6xy - 3x - 3y + 2 = 0,$$

and $$xy = -\tfrac{1}{12};$$

$$p = \frac{1 - 2x}{2(y - x)}, \quad q = \frac{2y - 1}{2(y - x)}.$$

76. Prof. Echols gave as the condition of a two-term formula

$$x'^2 + x'y' + y'^2 - \tfrac{3}{4}(x' + y') = 0,$$

where $x'h = a$, and $y'h = H - x'h$.

If we impose the condition that the sections (for convenience) are between bases, this formula becomes

$$x'^2 - x'y' + y'^2 - \tfrac{3}{4}(y' - x') = 0. \quad . \quad . \quad . \quad . \quad . \quad \text{(T)}$$

But $$H = h(y' + x');$$

$$a = x'h = xH;$$

$$y'h = H - xH = H(1 - x);$$

$$h = yH - xH = H(y - x) = \frac{H}{y' + x'}.$$

$$\therefore x' = \frac{x}{y - x}$$

and $$y' = \frac{1 - x}{y - x};$$

$$x' + y' = \frac{1}{y - x}.$$

From equation (25) we have

$$y = \frac{3x - 2}{6x - 3}.$$

But $h = (y - h)H = \dfrac{6x - 6x^2 - 2}{6x - 3}H.$

Substituting the values of x', y' in (T) and reducing, we get

$$6xy - 3x - 3y + 2 = 0.$$

77. The distance (h) apart of the sections S_1 and S_2 is variable. For maximum and minimum values of h we have

$$\frac{dh}{dx} = \frac{36x - 36x^2 - 6}{(6x - 3)} = 0.$$

For maximum value of h, $x = \frac{1}{2}$ and h = infinity.

For minimum value $x = (3 \pm \frac{1}{6}\sqrt{3})$ and $h = .5776H = \frac{H}{\sqrt{3}}$.

For real sections of the prismoid h varies from .5776 to .6666. The coordinates of the vertices of the equilateral hyperbolas give the sections that are nearest together. The height (h) is the ordinate between line OD and curve 123 in Fig. 43.

From Holditch's theorem we have

$$S = xB_2 + yB_1 - xyA.$$

Combine this with (3) by eliminating (A), and we get

$$M = B_1\frac{(3x-1)}{6x} + \frac{S}{6x(1-x)} + \frac{B_2(2-3x)}{6(1-x)}.$$

78. To find an expression for the mean area of the prismoid in terms of the mid-section and two sections S_1 and S_2 equidistant from the mid-section:

From (12) we have

$$M = B_1\frac{3x-1}{6x} + \frac{S_2}{6x(1-x)} + B_2\frac{2-3x}{6(1-x)}.$$

Then, again,

$$M = B_1\frac{2-3x}{6(1-x)} + \frac{S_1}{6x(1-x)} + B_2\frac{3x-1}{6x}.$$

But $$B_2 = 6M - B_1 - 4S_{\frac{1}{2}}.$$

Substitute this value of B_1 in the preceding equations, and then eliminate B_2, and we have

$$M = \frac{S_1 + S_2}{6(4x^2 - 4x + 1)} + \frac{6(4x^2 - 4x + 1)}{6(4x - 4x + 1)}S_{\frac{1}{2}}. \quad . \quad . \quad (26)$$

$$= k(S_1 + S_2) + jS.$$

The following table of coefficients is readily calculated:

x	k	j	x	k	j
0.000	.166	.666	.5000	.....	.....
.100	.260	.479	.6670	1.500	−1.333
.1456	.333	.333	.7041	1.000	−1.000
.2000	.463	.019	.7500	.667	− .333
.2114	.500	.000	.7886	.500	−0.000
.2500	.667	.333	.8564	.333	.333
.2959	1.000	−1.000	1.0000	.167	.667
.3333	1.509	−1.333			

Let OB = altitude of prismoid = 1. Then the two coefficient curves (k) and (j) can be plotted as in Fig. 44.

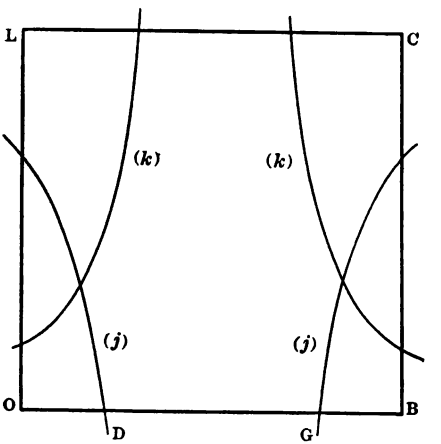

FIG. 44.

If $x = 0$ or 1, we get

$$M = \tfrac{1}{6}(B_1 + B_2 + 4S_{\frac{1}{2}}).$$

If $x = \dfrac{2 \pm \sqrt{2}}{2} = .1464$ or $.8536$, we get

$$M = \tfrac{1}{8}(S_1 + S_2 + S_{\frac{1}{2}}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If $x = \dfrac{3 \pm \sqrt{3}}{6} = OD$ or OG, we get Echols' formula

$$M = \tfrac{1}{2}(S_1 + S_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

79. If in formula (23) we make $x = \frac{1}{2} + z$, and $y = \frac{1}{2} - z$, the two sections S_1 and S_2 are equidistant from the mid-section.

$$\therefore M = \tfrac{1}{2}(S_1 + S_2) + (\tfrac{1}{12} - z^2)A. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

If $z = 0$, we get Koppe's formula, where

$$M = S_{\frac{1}{2}} + \tfrac{1}{12}A = S + \tfrac{1}{6}T.$$

If $z = \pm \frac{1}{6}$, we get Hirsch's formula, where

$$M = \tfrac{1}{2}(B_1 + B_2) - \tfrac{1}{6}A.$$

If $z = \pm \frac{1}{6}\sqrt{3}$, we get Echols' formula, where

$$M = \tfrac{1}{2}(S_1 + S_2).$$

We thus see that the Koppe, Hirsch, and Echols formulæ belong to one group or family.

The parabola QRL, Fig. 45, is the coefficient curve for A, where $OD = 1$, $RG = \frac{1}{12}$, $QO = DL = -\frac{1}{6}$, and $OE = DN = \frac{1}{6}(3 - \sqrt{3})$. The line KH is the coefficient curve for $(S_1 + S_2)$, where $OC = DH = \frac{1}{2}$.

If $z = \pm \frac{2}{3}\sqrt{21}$, we get another neat formula where

$$M = \tfrac{1}{2}(S_1 + S_2 - A). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

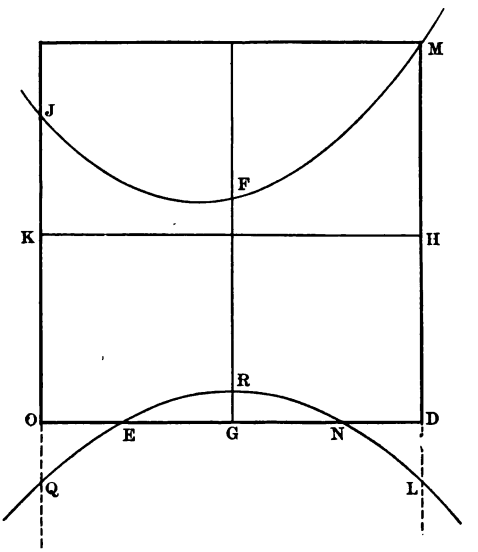

FIG. 45.

If $z = \pm \frac{1}{12}\sqrt{30}$, we get

$$M = \tfrac{1}{2}(S_1 + S_2 - T). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

80. If the section is a cubic function of its distance from any reference section, as

$$S_x = a + bx + cx^2 + dx^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (U)$$

there is only one two-term formula.

Let S_1 = section at a distance zH from B_1,

and S_2 = section at a distance yH from B_1,

q = coefficient of S_1,

p = coefficient of S_2.

Then for "two-term" formulæ the mean area (M) must fulfill the following conditions:

$$M = qS_1 + pS_2;$$

$$V = HM = H(qS_1 + pS_2).$$

But from Article 23 we have

$$V = H\left(a + \frac{bH}{2} + \frac{cH^2}{3} + \frac{dH^3}{4}\right).$$

$$\therefore\ M = a + \frac{bH}{2} + \frac{cH^2}{3} + \frac{dH^3}{4}.$$

Now in (U) make $a = zH$ and yH, and we get

$$S_1 = a + zbH + cz^2H^2 + dz^3H^3;$$

$$S_2 = a + byH + cy^2H^2 + dy^3H^3.$$

$$\therefore\ M = qS_1 + pS_2 = (q + p)a + bH(qz + py) + cH^2(qz^2 + py^2) + cH^3(qz^3 + py^3)$$

$$= a + \frac{bH}{2} + \frac{cH^2}{3} + \frac{dH^3}{4}.$$

Then for a two-term formula we must have

$$q + p = 1; \quad \ldots\ldots\ldots \text{(a)}$$

$$qz + py = \tfrac{1}{2}; \quad \ldots\ldots\ldots \text{(b)}$$

$$qz^2 + py^2 = \tfrac{1}{3}; \quad \ldots\ldots\ldots \text{(c)}$$

$$qz^3 + py^3 = \tfrac{1}{4}. \quad \ldots\ldots\ldots \text{(d)}$$

$$q = 1 - p = \frac{2y - 1}{2(y - z)} \quad \text{and} \quad p = \frac{1 - 2z}{2(y - z)}.$$

Substituting these values of p and q in (c) and (d) and reducing, we get

$$6yz^2 - 6y^2z - 3z^2 + 3y^2 = 2(y - z),$$

or $$3y + 3z - 6yz = 2, \quad \therefore\ y + z = \tfrac{1}{3}(2 + 6yz); \quad \ldots \text{(25)}$$

and $$8yz^3 - 4z^3 + 4y^3 - 8zy^3 = 2(y - z),$$

or $$4yz(z^2 - y^2) - 2(z^3 - y^3) = (y - z);$$

$$4yz(z + y) - 2(z + y) + 2yz = -1;$$

$$\tfrac{4}{3}yz(2 + 6yz) - \tfrac{8}{3}(2 + 6yz) + 2yz = -1; \quad \therefore\ yz = \tfrac{1}{6}.$$

$$\therefore\ y = \tfrac{1}{6}(3 \mp \sqrt{3}) \quad \text{and} \quad z = \tfrac{1}{6}(3 \pm \sqrt{3}).$$

There is, therefore, only one two-term formula where the section is a cubic function of its distance from some reference parallel section. If $d = 0$, we have an indefinite number of two-term formulæ as shown in Article 69

81. To find any section graphically:

By equating equation (22) with that of Newton and simplifying, we get

$$S_x = B_1 - x(3B_1 + B_2 - 4S_{\frac{1}{2}}) + x^2(2B_1 + 2B_2 - 4S_{\frac{1}{2}}). \quad \ldots \text{(32)}$$

This equation represents a parabola, and any section can be found graphically.

When $x = 0, \quad S = B_1;$
$x = .50, \quad S = S_{\frac{1}{2}};$
$x = 1, \quad S = B_2.$

In Fig. 45 lay off $OJ = B_1$, $FG = S_{\frac{1}{2}}$, and $DM = B_2$; then describe a parabola through these points. Any section at distance xH from the base can thus be found by measuring the proper ordinate.

For another proof of this see Grunert's Archiv, Vol. XXXIX.

(The two parabolas in Fig. 45 have no connection whatever.)

The area of $ODMFJ$, where $OD = H$, gives the volume of the prismoid.

If in (32) we make $S_x = M = \frac{1}{6}(B_1 + B_2 + 4S_{\frac{1}{2}})$, we get the value of x where the sections are equal to the mean area. It is thus possible to get a one-term formula.

If $P = 3B_1 + B_2 - 4S_{\frac{1}{2}}$, and $R = 2B_1 + 2B_2 - 4S_{\frac{1}{2}}$, we have

$$S = M = \tfrac{1}{6}(B_1 + B_2 + S_{\frac{1}{2}}) = B_1 + Px + Rx^2,$$

or $$Rx^2 - Px = \tfrac{1}{6}(B_2 + 4S_{\frac{1}{2}} - 5B_1).$$

By solving this and substituting the roots x_1 and x_2 in (32), we get M.

If in (32) we make $S_x = M = \frac{1}{2}(B_1 + B_2) - \frac{1}{6}A$ and make $z = 1 - x$, we get

$$S_x = \tfrac{1}{2}(B_1 + B_2) - \tfrac{1}{6}A = B + x(B_2 - B_1 - A) + Ax^2.$$

This equation will also give the values of x where the section $S_x = M$. In the same way we can get a one-term formula from this associate as we did from (42).

We find that

$$x = \frac{1}{2A}(A + B_1 - B_2) \pm \left[\frac{3(B_1 - B_2)^2 + A^2}{12A}\right]^{\frac{1}{2}}.$$

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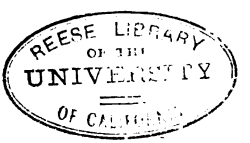

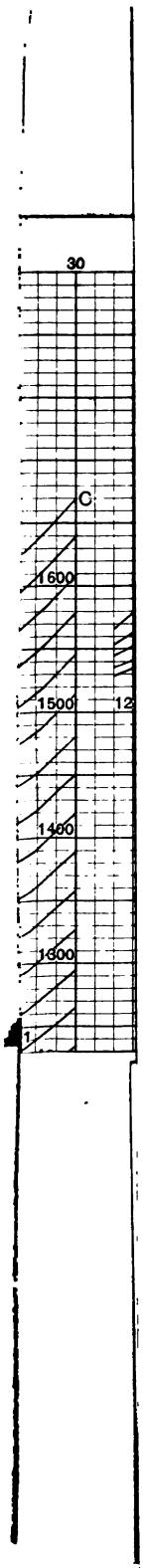
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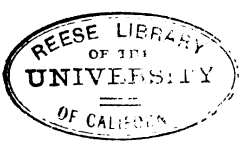
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